

EECE 301
Signals & Systems
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Note Set #25

- D-T Systems: Partial Fraction Expansion for ZT

The idea of PFE is to break a complicated ratio of two polynomials into a sum of simpler terms:

$$\frac{3z^2 - z}{z^2 - 3z + 2} = \frac{5z}{z - 2} + \frac{-2z}{z - 1}$$

Going the other way is easy and obvious:

$$\begin{aligned} \frac{5z}{z - 2} + \frac{-2z}{z - 1} &= \frac{5z(z - 1)}{(z - 2)(z - 1)} + \frac{-2z(z - 2)}{(z - 2)(z - 1)} \\ &= \frac{5z(z - 1) - 2z(z - 2)}{(z - 2)(z - 1)} = \frac{3z^2 - z}{z^2 - 3z + 2} \end{aligned}$$

Notice that the denominator of the original is the product of the denominators of the two terms... thus, we see that finding the PFE involves factoring the denominator of the original function!

Note: when applying PFE for the ZT it is usually helpful to first divide the function to be expanded by z... then expand... then “undivide” by z.

Exactly how we do the expansion depends on the structure of the original function that is to be expanded:

- The first characteristic structure is if the denominator has or does not have any repeated roots.
- The second characteristic structure is if the function has or does not have any so-called “direct terms”.
 - There are no direct terms when the order of the numerator of the function is strictly less than the order of the denominator.
 - Let p be the difference between the numerator order and denominator order. Then there are $p+1$ direct terms.

Doing PFE by hand for the case of real roots, no repeated roots, and no direct terms is quite easy.

For the more complicated cases the use of MATLAB is encouraged.

Ex. #1: No Direct Terms, No Repeated Roots, No Complex Roots

$$Y(z) = \frac{3z + 1}{z^2 + 3z + 2}$$

Function to expand

First Step: Divide by z ... $Y(z) / z \triangleq G(z) = \frac{3z + 1}{z(z^2 + 3z + 2)}$

Second Step: Any Direct Terms? No... (Num Order = 1) < (Denom Order = 3)

Third Step: Any Repeated Roots? No... Denom = $z(z+2)(z+1)$

Fourth Step: Write in expanded form with one term for each denom factor

$$G(z) = \frac{3z + 1}{z(z + 2)(z + 1)} = \frac{r_1}{z + 2} + \frac{r_2}{z + 1} + \frac{r_3}{z}$$

“Residuals”

In general...

$$G(z) = \frac{r_1}{z + p_1} + \frac{r_2}{z + p_2} + \dots + \frac{r_N}{z + p_N}$$

Fifth Step: Determine the residuals

$$r_i = (z + p_i) G(z) \Big|_{z=-p_i}$$

$$G(z) = \frac{3z + 1}{z(z + 2)(z + 1)}$$

$$r_1 = (z + 2) G(z) \Big|_{z=-2} = \frac{3z + 1}{z(z + 1)} \Big|_{z=-2} = \frac{3(-2) + 1}{-2(-2 + 1)} = \frac{-5}{2}$$

$$r_2 = (z + 1) G(z) \Big|_{z=-1} = \frac{3z + 1}{z(z + 2)} \Big|_{z=-1} = \frac{3(-1) + 1}{-1(-1 + 2)} = 2$$

$$r_3 = (z) G(z) \Big|_{z=0} = \frac{3z + 1}{(z + 1)(z + 2)} \Big|_{z=0} = \frac{3(0) + 1}{(0 + 1)(0 + 2)} = \frac{1}{2}$$

Sixth Step: Sub in residuals and multiply all terms by z (assuming you pre-divided by z):

$$Y(z) = zG(z) = \frac{\frac{-5}{2} z}{z + 2} + \frac{2z}{z + 1} + \frac{\frac{1}{2} z}{z}$$

For comparison... re-do this example using MATLAB:

$$Y(z)/z = \frac{3z + 1}{z(z^2 + 3z + 2)} = \frac{3z + 1}{z^3 + 3z^2 + 2z}$$

```
>> [r,p,k]=residue([3 1],[1 3 2 0])
```

r =

-2.5000
2.0000
0.5000

Residuals

p =

-2
-1
0

Roots

k =

[]

Direct

(Empty vector → none)

$$Y(z) = zG(z) = \frac{-\frac{5}{2}z}{z+2} + \frac{2z}{z+1} + \frac{\cancel{\frac{1}{2}z}}{z}$$

Ex. #2: One Direct Term, No Repeated Roots, No Complex Roots

$$Y(z)/z = \frac{2z^2 + 3z - 1}{z^2 - 3z + 2}$$

>> [r,p,k]=residue([2 3 -1],[1 -3 2])

r =		}	$Y(z)/z = \frac{13}{z-2} + \frac{-4}{z-1} + 2$
13	Residuals		
-4			
p =		}	
2	Roots		
1			
k =		}	$\Rightarrow Y(z) = \frac{13z}{z-2} + \frac{-4z}{z-1} + 2z$
2	Direct		

Ex. #3: Two Direct Terms, No Repeated Roots, No Complex Roots

$$Y(z) / z = \frac{7z^3 + 2z^2 + 3z - 1}{z^2 - 3z + 2}$$

>> [r,p,k]=residue([7 2 3 -1],[1 -3 2])

r =

69

-11

p =


2

1

k =

7 23

$$Y(z) / z = \frac{69}{z-2} + \frac{-11}{z-1} + 7z + 23$$


$$Y(z) = \frac{69z}{z-2} + \frac{-11z}{z-1} + 7z^2 + 23z$$

Note: if there were M elements in vector k the left most term would be the coefficient of a z^{M-1} order direct term. The others are coefficients of descending powers of z

Ex. #4: No Direct Terms, A Double Root, No Complex Roots

$$Y(z)/z = \frac{2z^2 + 3z - 1}{z^3 - 5z^2 + 8z - 4} = \frac{2z^2 + 3z - 1}{(z-1)(z-2)^2}$$

>> [r,p,k]=residue([2 3 -1],[1 -5 8 -4])

r =

-2.0000
13.0000
4.0000

$$Y(z)/z = \frac{-2}{z-2} + \frac{13}{(z-2)^2} + \frac{4}{z-1}$$

p =

2.0000
2.0000
1.0000



$$Y(z) = \frac{-2z}{z-2} + \frac{13z}{(z-2)^2} + \frac{4z}{z-1}$$

k =

[]

Note: Repeated Roots give rise to terms with increasing powers up to the “order” of the repeated root. (Here we have a double root... so we go up to power of 2).

Ex. #5: No Direct Terms, No Repeated Roots, Two Complex Roots

$$Y(z)/z = \frac{2z^2 + 3z - 1}{z^3 - 3z^2 + 4z - 2} = \frac{2z^2 + 3z - 1}{(z-1)(z^2 - 2z + 2)}$$

Has complex roots

>> [r,p,k]=residue([2 3 -1],[1 -3 4 -2])

r =
 -1.0000 - 3.5000i complex
 -1.0000 + 3.5000i residues...
 4.0000 are
 conjugates

p =
 1.0000 + 1.0000i complex
 1.0000 - 1.0000i roots...
 1.0000 are
 conjugates

k =
 []

$$Y(z)/z = \frac{-1 - j3.5}{z - (1 + j)} + \frac{-1 + j3.5}{z - (1 - j)} + \frac{4}{z - 1}$$



$$Y(z) = \frac{3.64e^{-j1.85}z}{z - (\sqrt{2}e^{j\pi/4})} + \frac{3.64e^{-j1.85}z}{z - (\sqrt{2}e^{-j\pi/4})} + \frac{4z}{z - 1}$$