BINGHAMTON UNIVERSITY

State University of New York

EECE 301
Signals & Systems
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Note Set #24

- D-T Systems: Z-Transform … "Power Tool" for system analysis

Z-Transform & D-T Systems

Z-Transform is a powerful tool for the analysis and design of DT LTI Systems

Z-T is used to

Solve difference equations with non-zero initial conditions

We’ll do this later

Characterize systems using the “Transfer Function”

Our initial focus is here

Z-transform definitions

Given a D-T signal $x[n]$ $-\infty < n < \infty$ we've already seen how to use the DTFT:

$$DTFT : X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

Periodic in Ω with period 2π

Unfortunately the DTFT doesn't "converge" for some signals… the ZT mitigates this problem by including decay in the transform:

$$e^{-j\Omega n} \quad vs. \quad \alpha^{-n}e^{-j\Omega n} \equiv (\alpha e^{j\Omega})^{-n} \equiv z^{-n}$$

Controls decay of summand

For the Z-transform we use: $z = \alpha e^{j\Omega}$. So… z is just a complex variable that we almost always view in polar form

$$DTFT : X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} \Rightarrow ZT : X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

There are two forms of the Z-Transform:

Two-Sided Z-transform ("Bilateral" ZT)

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \qquad z \text{ is complex-valued}$$

Our Initial Focus is Here

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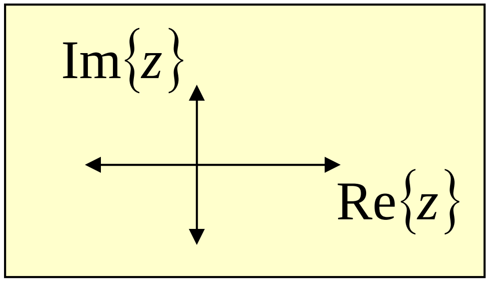
We'll use this later

If $x[n]$ is a causal signal then these two types of ZT are identical.

"Causal Signal" means that $x[n] = 0$ for $n < 0$

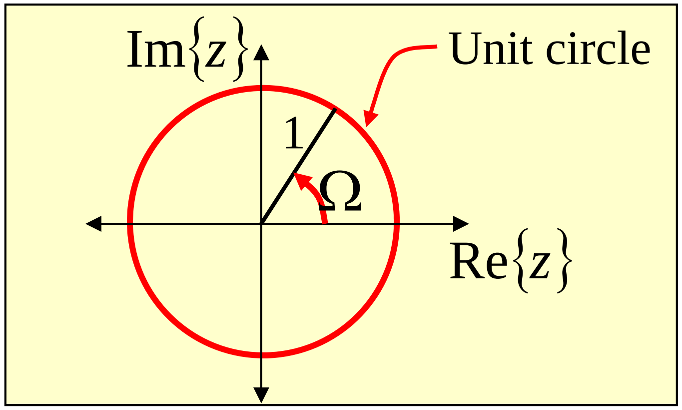
So… the Z-Transform gives a complex-valued function on the "z-plane"



For the Z-Transform we'll need to divide the plane into two parts:

- those values of *z* inside the "unit circle"
- those values of *z* outside the "unit circle"

"**<u>Unit Circle</u>**" = all *z* such that $|z| = 1$, i.e. all $z = e^{j\Omega}$



Region of Convergence (ROC)

Set of all z values for which the sum in the ZT definition converges

Each signal has its own region of convergence.

Example of Finding the ZT: Unit Impulse Sequence

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

$$Z\{\delta[n]\} = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n}$$

$$= 1 \times z^{0} + 0 \times z^{-1} + 0 \times z^{-2} + \cdots$$

$$= 1$$

$$\delta[n] \quad \leftrightarrow \quad 1$$

ROC = all complex #'s

This result and many others are on the Table of Z Transforms

Example of Finding the ZT: Unit Step $u[n]$

$$U(z) = \sum_{n=-\infty}^{\infty} u[n] z^{-n} = \sum_{n=0}^{\infty} z^{-n} = \frac{z}{z-1} = \frac{1}{1-z^{-1}}$$

ROC = all z such that $|z| > 1$

Using standard result for "geometric sum"

$$u[n] \quad \leftrightarrow \quad \frac{z}{z-1} = \frac{1}{1-z^{-1}}$$

Example of Finding the ZT: Causal Exponential

$$x[n] = a^n u[n]$$

Again using geometric sum: $X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n = \frac{z}{z-a} = \frac{1}{1-az^{-1}}$

ROC = all z such that $|z| > |a|$

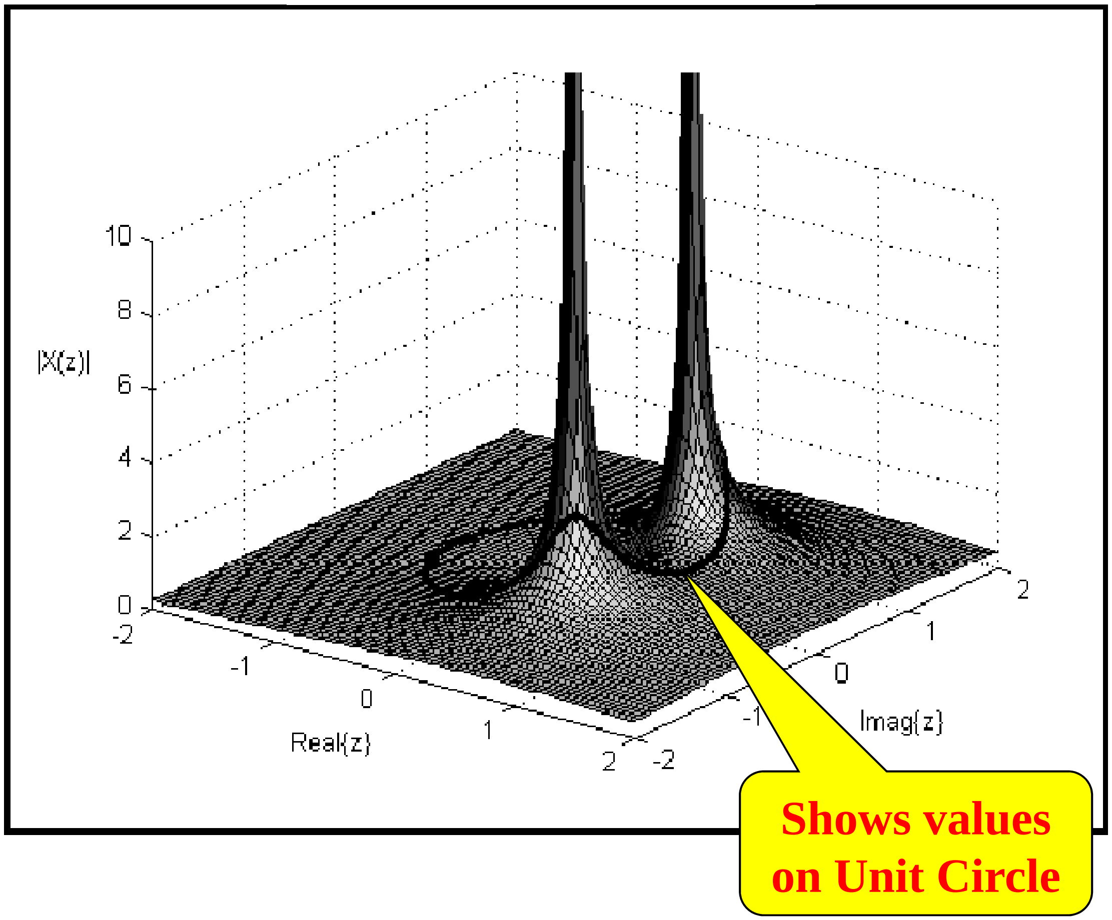
$$a^n u[n] \quad \leftrightarrow \quad \frac{z}{z-a} = \frac{1}{1-az^{-1}}$$

Relationship between ZT & DTFT

If ROC includes the unit circle, then we can say that: $X(\Omega) = X(z)\big|_{z=e^{j\Omega}}$

$X(\Omega)$ = "walk around the unit circle" and get $X(z)$ values

Surface Plot of $|X(z)|$



Explains why $X(\Omega)$ is periodic… Ω is an "angle around the unit circle"

⇒ Once we've walked around the unit circle… going farther just repeats the values $X(z)$ that we are grabbing

Inverse Z-T

There is an integral inversion formula but it is not really used in practice!

$\Rightarrow$ Use Partial Fraction Expansion (PFE)

The use of PFE here is ***almost*** exactly the same as for Laplace transforms that you may have seen before (and we'll see later).

… the only difference is that you first divide by *z* ***before*** performing the PFE… then after expanding you multiply by *z* to get the final expansion.

Example of Partial Fraction for Inverse ZT:

See Next Note Set for details on PFE

Suppose you want to find the inverse ZT of

$$Y(z) = \frac{z+1}{z^2 + \frac{3}{4}z + \frac{1}{8}}$$

First divide $Y(z)$ by z to get:

$$\frac{Y(z)}{z} = \frac{z+1}{z^3 + \frac{3}{4}z^2 + \frac{1}{8}z}$$

Then use matlab's residue to do a partial fraction expansion on $Y(z)/z$

```
[r,p,k]=residue([1 1],[1 0.75 0.125 0])

r =                p =                k = []
  4                  -0.5000
 -12                 -0.2500
  8                   0
```

For each term:

$$\frac{r}{z-p}$$

$$\frac{Y(z)}{z} = \frac{4}{z+\frac{1}{2}} + \frac{-12}{z+\frac{1}{4}} + \frac{8}{z}$$

$$Y(z) = \frac{4z}{z+\frac{1}{2}} - \frac{12z}{z+\frac{1}{4}} + 8$$

Now… the point of dividing by z becomes clear… you get terms like this (with z's in the numerator)… and they are on the ZT table!!!

$$y[n] = 4(-\tfrac{1}{2})^n u[n] - 12(-\tfrac{1}{4})^n u[n] + 8\delta[n]$$

A Few Properties of Bilateral ZT

There are several other properties… they are listed on the Table of Z Transform Properties.

Linearity: Same ideas as for CTFT and DTFT

Time Shift

$$\text{If } x[n] \leftrightarrow X(z), \quad \text{then} \quad x[n-q] \leftrightarrow z^{-q}X(z)$$

Note: Here q can be positive or negative

$$\text{"Proof"}: \quad X(z) = \cdots + x[-2]z^2 + x[-1]z^1 + x[0]z^0 + x[1]z^{-1} + x[2]z^{-2} + \cdots$$

$$Z\{x[n-2]\} = \cdots + x[-1-2]z^1 + x[0-2]z^0 + x[1-2]z^{-1} + x[2-2]z^{-2} + x[3-2]z^{-3} + \ldots$$

$$= \cdots + x[-3]z^1 + x[-2]z^0 + x[-1]z^{-1} + x[0]z^{-2} + x[1]z^{-3} + \ldots$$

$$= \cdots + x[-3]z^{-2}z^3 + x[-2]z^{-2}z^2 + x[-1]z^{-2}z^1 + x[0]z^{-2}z^0 + x[1]z^{-2}z^{-1} + \ldots$$

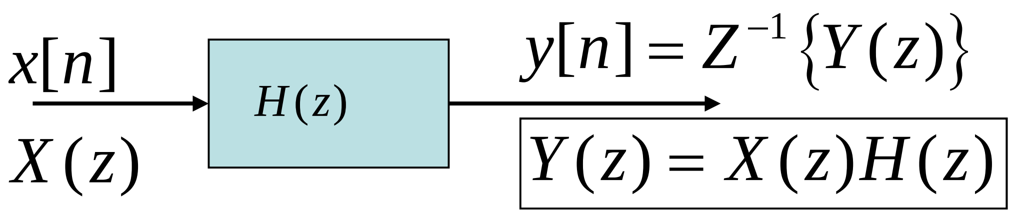
Pull out the z^{-2}

$$= z^{-2}\underbrace{\left[\cdots + x[-3]z^3 + x[-2]z^2 + x[-1]z^1 + x[0]z^0 + x[1]z^{-1} + \ldots\right]}_{= X(z)}$$

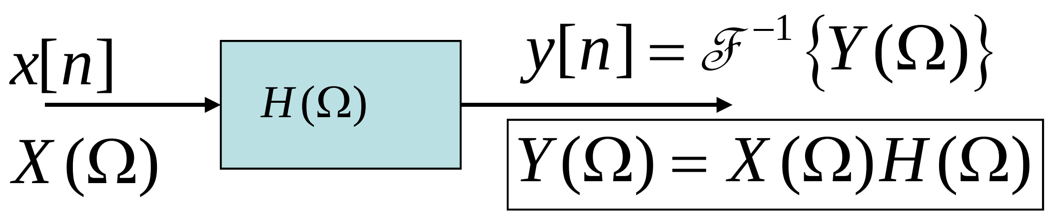
System Property

The output of a LTI DT system has ZT $Y(z)$ given by $\boxed{Y(z) = X(z)H(z)}$

So we have:



Note how similar this is to what we saw for DTFT:



Terminology

- Frequency Response: $H(\Omega)$
- Transfer Function: $H(z)$