

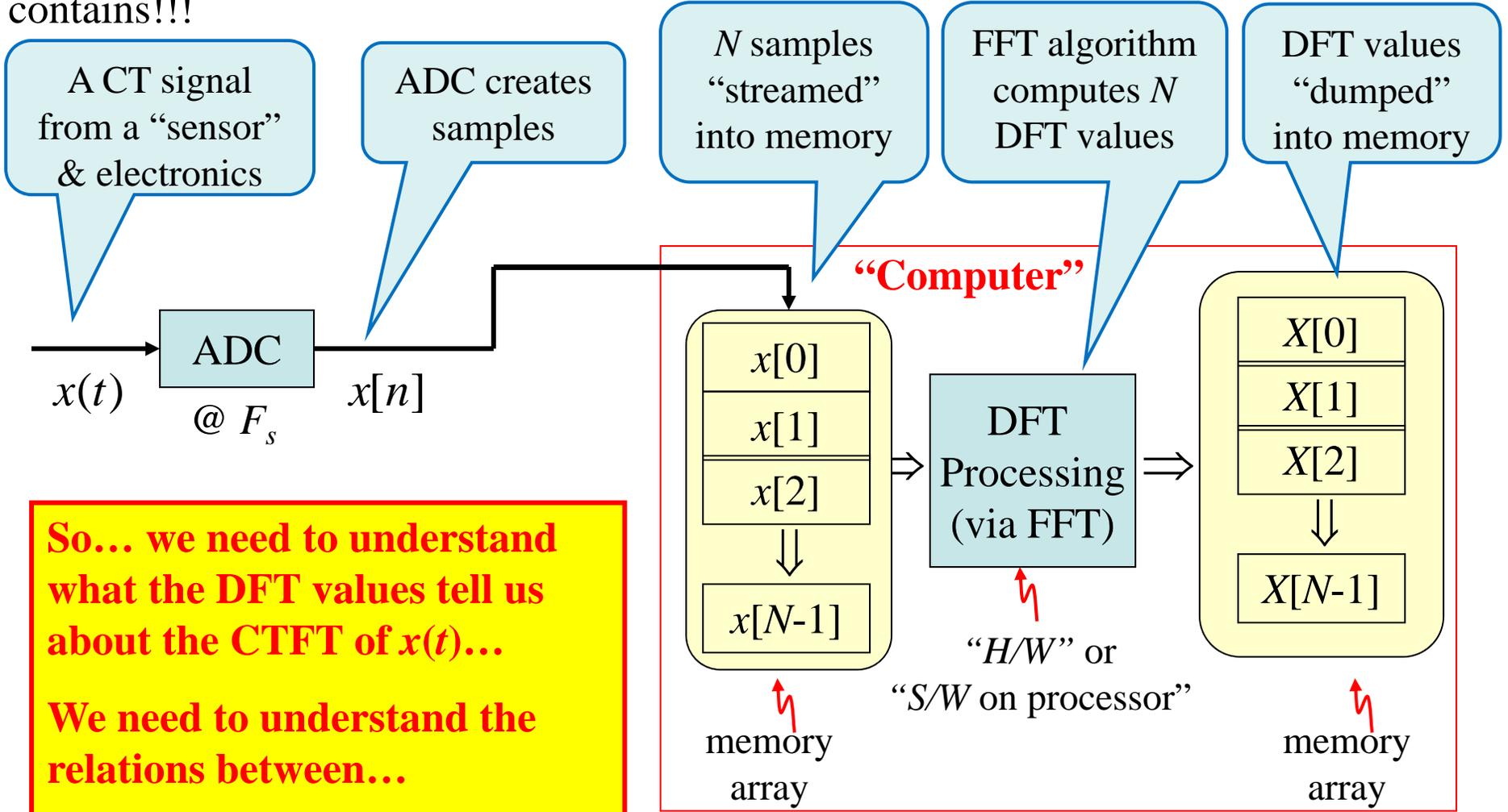
EECE 301  
Signals & Systems  
Prof. Mark Fowler

**Note Set #21**

- D-T Signals: Relation between DFT, DTFT, & CTFT

# We can use the DFT to implement numerical FT processing

This enables us to numerically analyze a signal to find out what frequencies it contains!!!



**So... we need to understand what the DFT values tell us about the CTFT of  $x(t)$ ...**

**We need to understand the relations between...**

**CTFT, DTFT, and DFT**

We'll mathematically explore the link between DTFT & DFT in **two cases**:

1. For  $x[n]$  of **finite duration**:

$$\dots 0 \ 0 \ x[0] \ x[1] \ x[2] \ \dots \ X[N-1] \ 0 \ 0$$

$N$  “non-zero” terms

(of course, we could have some of the interior values = 0)

For this case... we'll assume that the signal is zero outside the range that we have captured.

So... we have all of the meaningful signal data.

This case hardly ever happens... but it's easy to analyze and provides a perspective for the 2<sup>nd</sup> case

This is the practical case.

2. For  $x[n]$  of **infinite duration** ...or at least of duration longer than what we can get into our “DFT Processor” inside our “computer”.

So... we don't have all the meaningful signal data.

What effect does that have? How much data do we need for a given goal?

## DFT & DTFT: Finite Duration Case

If  $x[n] = 0$  for  $n < 0$  and  $n \geq N$  then the DTFT is:

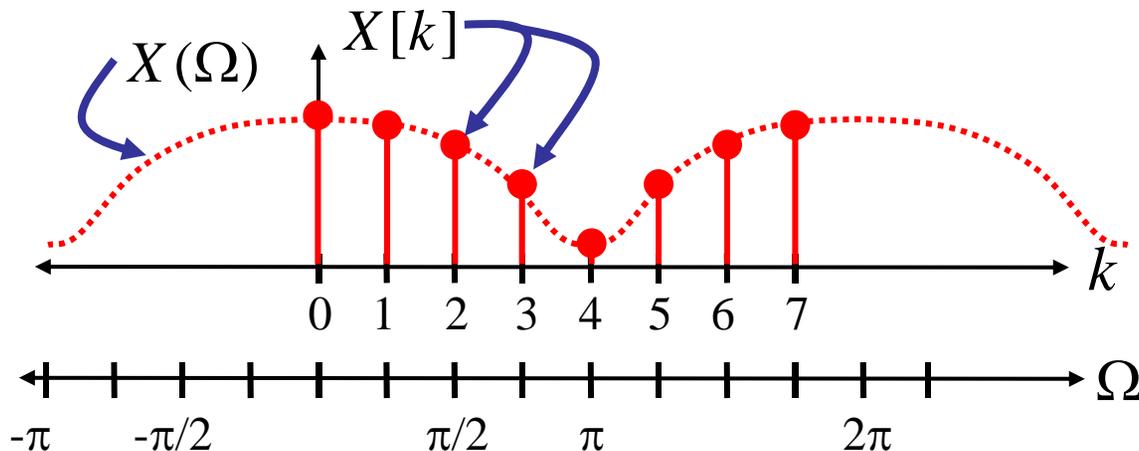
$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = \sum_{n=0}^{N-1} x[n]e^{-j\Omega n}$$

we can leave out terms that are zero

Now... if we take these  $N$  samples and compute the DFT (using the FFT, perhaps) we get:

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N} \quad k = 0, 1, 2, \dots, N-1$$

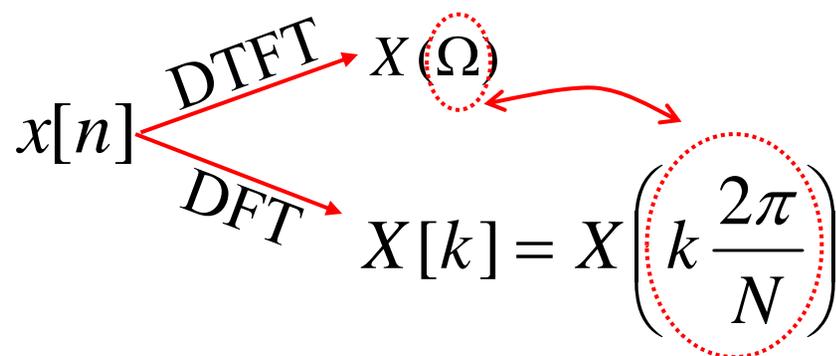
Comparing these we see that for the finite-duration signal case:  $X[k] = X\left(k \frac{2\pi}{N}\right)$



### DTFT & DFT :

DFT points lie exactly on the finite-duration signal's DTFT!!!

## Summary of DFT & DTFT for a *finite* duration $x[n]$



Points of DFT are “samples” of DTFT of  $x[n]$

The number of samples  $N$  sets how closely spaced these “samples” are on the DTFT... seems to be a limitation.

### “Zero-Padding Trick”

After we collect our  $N$  samples, we tack on some additional zeros at the end to trick the “DFT Processing” into thinking there are really more samples.

(Since these are zeros tacked on they don’t change the values in the DFT sums)

If we now have a total of  $N_Z$  “samples” (including the tacked on zeros), then the spacing between DFT points is  $2\pi/N_Z$  which is smaller than  $2\pi/N$

### Ex. 4.11 DTFT & DFT of pulse

$$x[n] = \begin{cases} 1, & n = 0, 1, 2, \dots, 2q \\ 0, & \text{otherwise} \end{cases} \quad \text{Recall: } p_q[n] = \begin{cases} 1, & n = -q, \dots, -1, 0, 1, \dots, q \\ 0, & \text{otherwise} \end{cases}$$

Then...  $x[n] = p_q[n - q]$

Note: we'll need the delay property for DTFT

From DTFT Table:  $p_q[n] \leftrightarrow P_q(\Omega) = \frac{\sin[(q + 0.5)\Omega]}{\sin[\Omega/2]}$

From DTFT Property Table  
(Delay Property):

$$X(\Omega) = \frac{\sin[(q + 0.5)\Omega]}{\sin[\Omega/2]} e^{-jq\Omega}$$

Since  $x[n]$  is a finite-duration signal then the DFT of the  $N = 2q + 1$  non-zero samples is just samples of the DTFT:

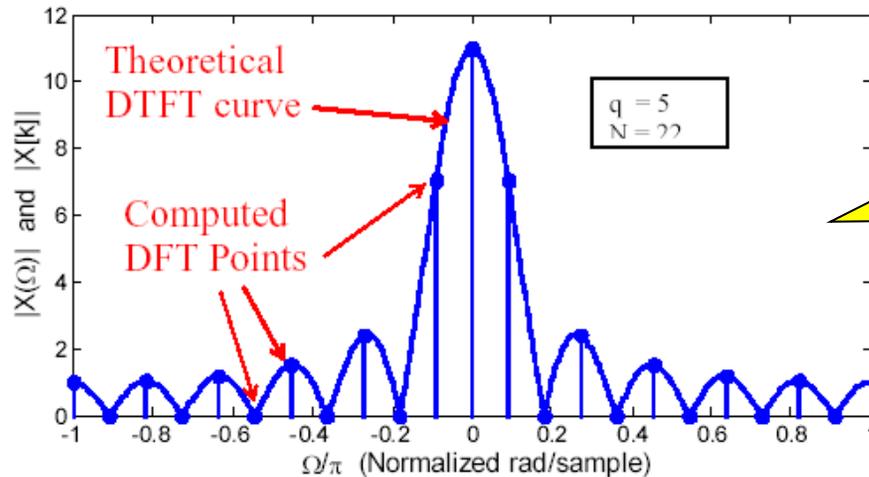
$$X[k] = X\left(k \frac{2\pi}{N}\right)$$

$$X[k] = \frac{\sin[(q + .5)2\pi k / N]}{\sin[\pi k / N]} e^{-jq2\pi k / N}$$

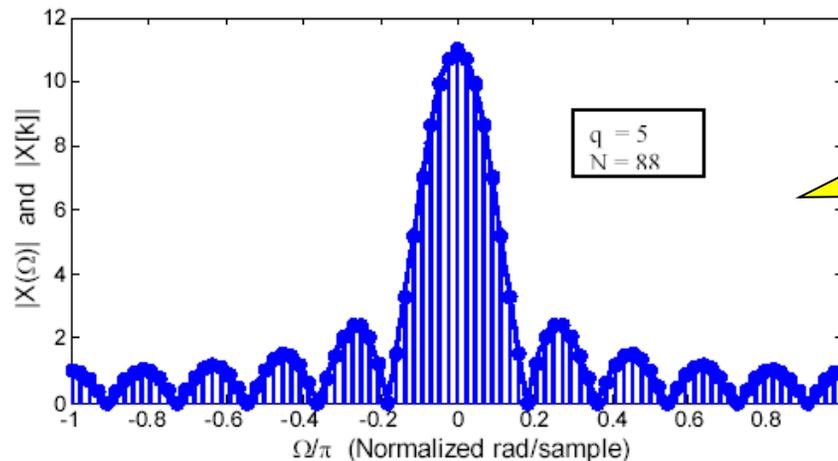
Note that if we don't zero pad, then all but the  $k = 0$  DFT values are zero!!!

That doesn't show what the DTFT looks like! So we need to use zero-padding.

Here are two numerically computed examples, both for the case of  $q = 5$ :



For the case of **zero-padding 11 zeros** onto the end of the signal... the DFT points still don't really show what the DTFT looks like!



For the case of **zero-padding 77 zeros** onto the end of the signal... **NOW** the DFT points really show what the DTFT looks like!

DFTs were computed using matlab's fft command... see code on next slide

```

omega=eps+(-1:0.0001:1)*pi;
q=5; % used to set pulse length to 11 points
X=sin((q+0.5)*omega)./sin(omega/2);
subplot(2,1,1)
plot(omega/pi,abs(X)); % plot magn of DTFT
xlabel('\Omega\pi (Normalized rad/sample)')
ylabel('|X(\Omega)| and |X[k]|')
hold on

```

Compute the DTFT  
Equation derived for the pulse. Using *eps* adds a very small number to avoid getting  $\Omega = 0$  and then dividing by 0

```

x=zeros(1,22); % Initially fill x with 22 zeros
x(1:(2*q+1))=1; % Then fill first 11 pts with ones

```

Make the zero-padded signal

```

Xk=fftshift(fft(x)); % fft computes the DFT and fftshift re-orders points
% to between -pi and pi

```

Compute the DFT

```

omega_k=(-11:10)*2*pi/16; % compute DFT frequencies, except make them
% between -pi and pi
stem(omega_k/pi,abs(Xk)); % plot DFT vs. normalized frequencies

```

Compute the DFT point's frequency values and plot the DFT

```

hold off
subplot(2,1,2)
plot(omega/pi,abs(X));
xlabel('\Omega\pi (Normalized rad/sample)')
ylabel('|X(\Omega)| and |X[k]|')
hold on
x=zeros(1,88);
x(1:(2*q+1))=1;
Xk=fftshift(fft(x));
omega_k=(-44:43)*2*pi/88;
stem(omega_k/pi,abs(Xk));
hold off

```

## Important Points for *Finite-Duration* Signal Case

- DFT points lie on the DTFT curve... perfect view of the DTFT
  - But... only if the DFT points are spaced closely enough
- Zero-Padding doesn't change the shape of the DFT...
- It just gives a denser set of DFT points... all of which lie on the true DTFT
  - Zero-padding provides a better view of this “perfect” view of the DTFT

## DFT & DTFT: Infinite Duration Case

As we said... in a computer we cannot deal with an infinite number of signal samples.

So say there is some signal that “goes on forever” (or at least continues on for longer than we can or are willing to grab samples)

$$x[n] \quad n = \dots, -3, -2, -1, 0, 1, 2, 3, \dots$$

We **only grab  $N$  samples**:  $x[n], n = 0, \dots, N - 1$     **We've lost some information!**

We can define an “imagined” finite-duration signal:

$$x_N[n] = \begin{cases} x[n], & n = 0, 1, 2, \dots, N - 1 \\ 0, & \textit{elsewhere} \end{cases}$$

We can compute the DFT of the  $N$  collected samples:

$$X_N[k] = \sum_{n=0}^{N-1} x_N[n] e^{-j2\pi nk/N} \quad k = 0, 1, \dots, N - 1$$

**Q: How does this DFT of the “truncated signal” relate to the “true” DTFT of the full-duration  $x[n]$ ? ...which is what we really want to see!!**

$$\text{"True" DTFT: } X_{\infty}(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

What we want to see

$$\begin{aligned} \text{DTFT of truncated signal: } X_N(\Omega) &= \sum_{n=-\infty}^{\infty} x_N[n]e^{-j\Omega n} \\ &= \sum_{n=0}^{N-1} x[n]e^{-j\Omega n} \end{aligned}$$

A distorted version of what we want to see

$$\text{DFT of collected signal data: } X_N[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$$

What we can see

DFT gives samples of  $X_N(\Omega)$

So... DFT of collected data gives “samples” of DTFT of truncated signal  
 $\neq$  “True” DTFT

$\Rightarrow$  DFT of collected data does not perfectly show DTFT of complete signal.

Instead, the DFT of the data shows the DTFT of the truncated signal...

So our goal is to understand what kinds of “errors” are in the “truncated” DTFT  
...then we’ll know what “errors” are in the computed DFT of the data

To see what the DFT does show we need to understand how

$X_N(\Omega)$  relates to  $X_\infty(\Omega)$

First, we note that:

$$x_N[n] = x[n] \underbrace{p_q[n-q]}_{\text{DTFT}} \rightarrow P_q(\Omega) = \frac{\sin[N\Omega/2]}{\sin[\Omega/2]} e^{-j(N-1)\Omega/2}$$

with  $N=2q+1$

From “mult. in time domain” property in DTFT Property Table:

$$X_N(\Omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_\infty(\lambda) P_q(\Omega - \lambda) d\lambda$$

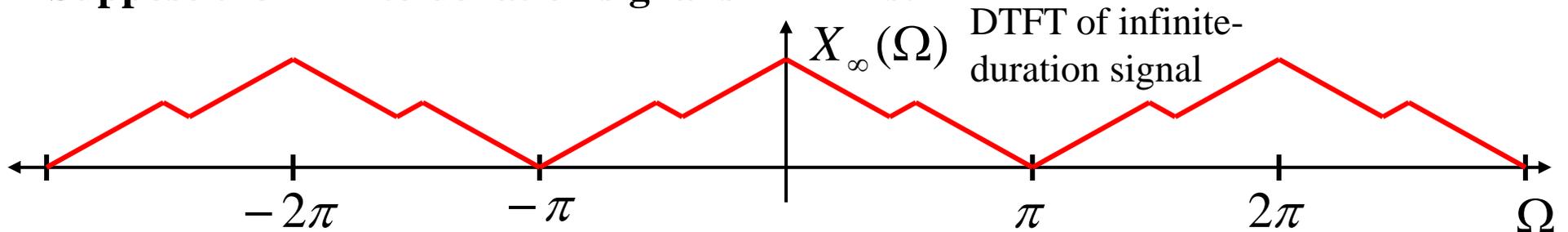
causes “smearing”  
of  $X_\infty(\Omega)$

$\Rightarrow$  So...  $X_N(\Omega)$  ...which we can see via the DFT  $X_N[k]$  ...

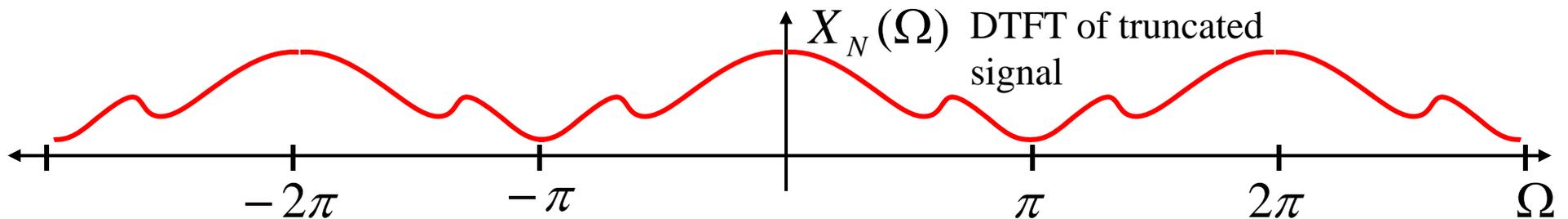
is a “smeared” version of  $X_\infty(\Omega)$

**“Fact”:** The more data you collect, the less smearing  
... because  $P_q(\Omega)$  becomes more like  $\delta(\Omega)$

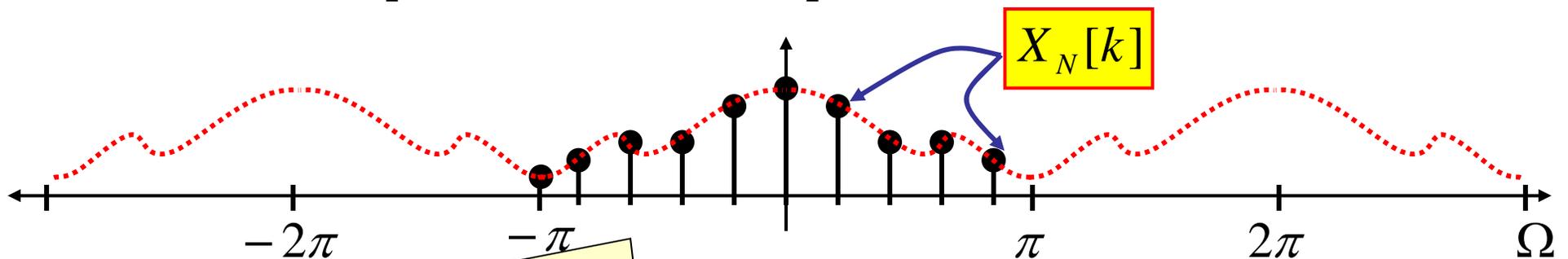
Suppose the infinite-duration signal's DTFT is:



Then it gets smeared into something that might look like this:



Then the DFT computed from the  $N$  data points is:



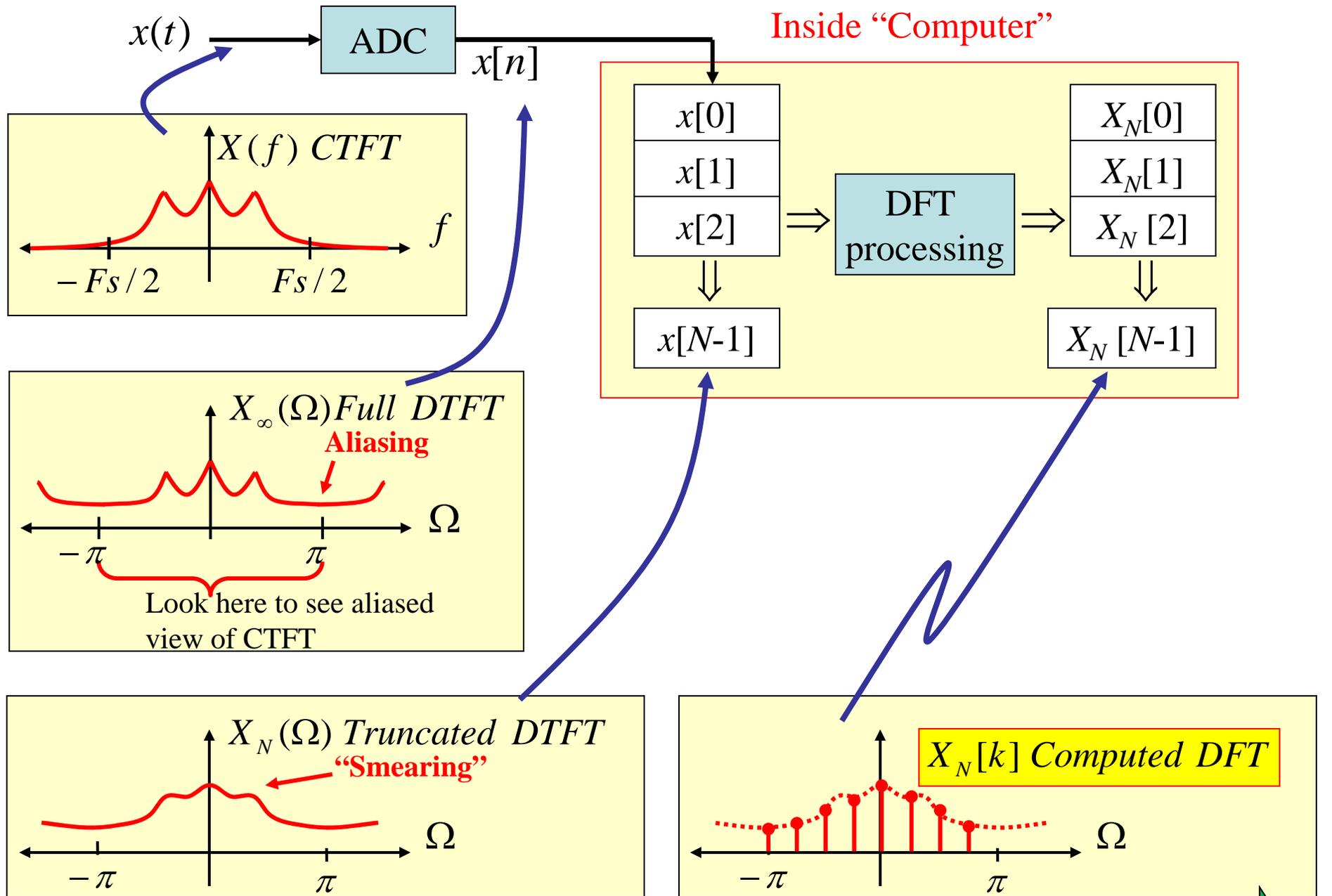
The DFT points are shown *after* “upper” points are moved (e.g., by MATLAB “fftshift”)

The only case that *really* happens in practice!

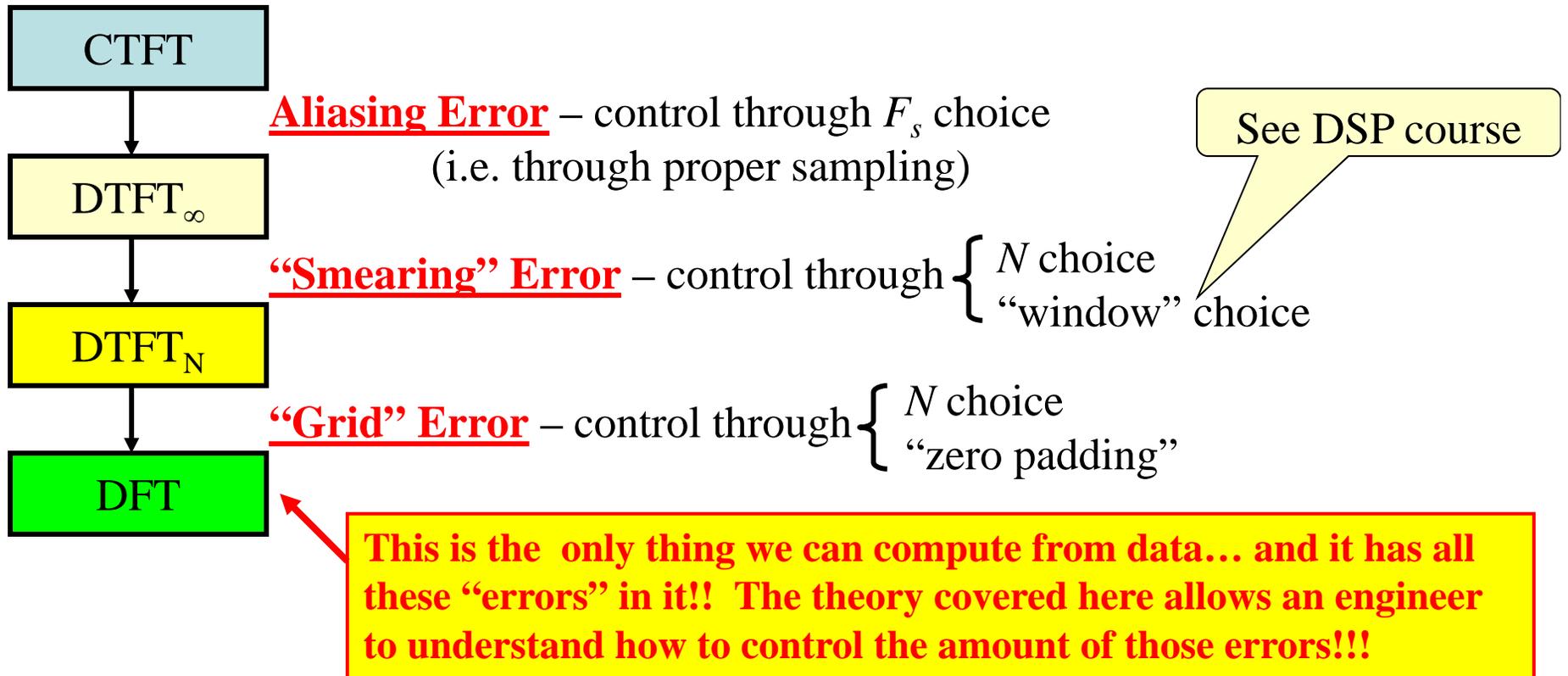
## Important points for Infinite-Duration Signal Case

1. DTFT of finite collected data is a “smeared” version of the DTFT of the infinite-duration data
2. The computed DFT points lie on the “smeared” DTFT curve... not the “true” DTFT
  - a. This gives an imperfect view of the true DTFT!
3. “Zero-padding” gives denser set of DFT points... a better view of this imperfect view of the desired DTFT!!!

# Connections between the CTFT, DTFT, & DFT



## Errors in a Computed DFT



### Zero padding trick

Collect  $N$  samples  $\rightarrow$  defines  $X_N(\Omega)$

Tack  $M$  zeros on at the end of the samples

Take  $(N + M)$ pt. DFT  $\rightarrow$  gives points on  $X_N(\Omega)$  spaced by  $2\pi/(N+M)$   
(rather than  $2\pi/N$ )

