Binghamton University
State University of New York

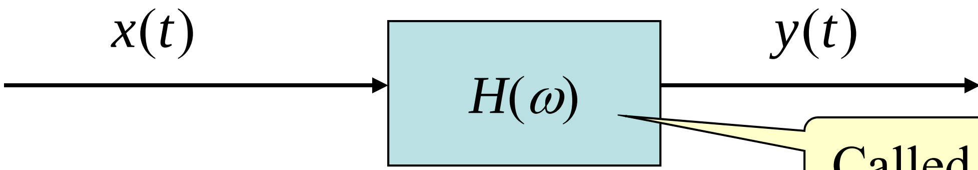
EECE 301
Signals & Systems
Prof. Mark Fowler

Note Set #15

- C-T Systems: CT Filters & Frequency Response

Ideal Filters

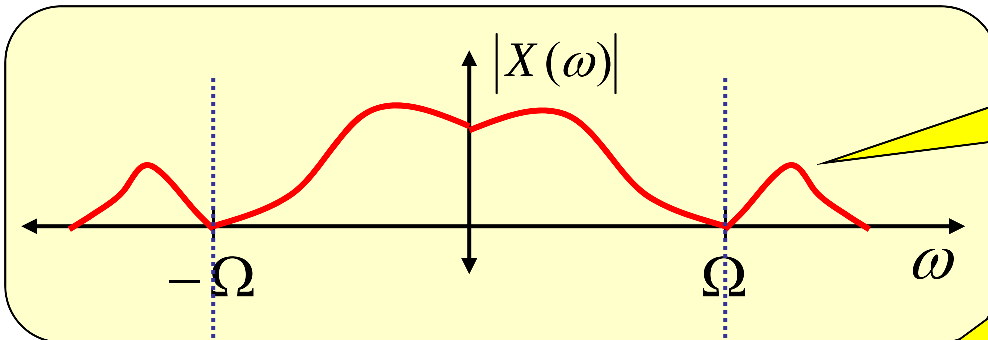
Often we have a scenario where part of the input signal's spectrum comprises "what we want" and part comprises something we "do not want". We can use a filter to remove (or filter out) the "bad part".



Called "a Filter" in this case

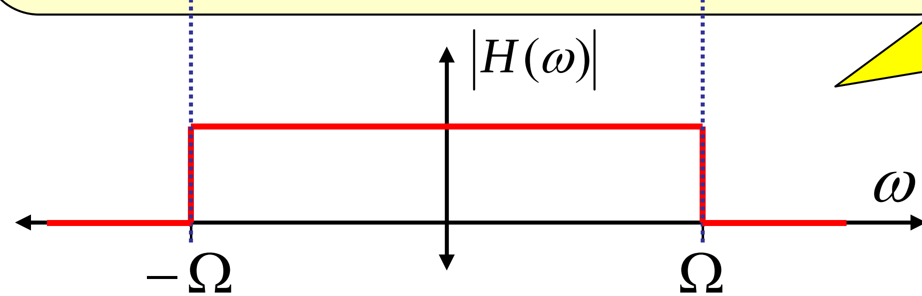
Case #1:

Spectrum of the Input Signal



Undesired Part

In this case, we want a filter like this:



A filter that "passes" low frequencies is called a **"low-pass filter"**

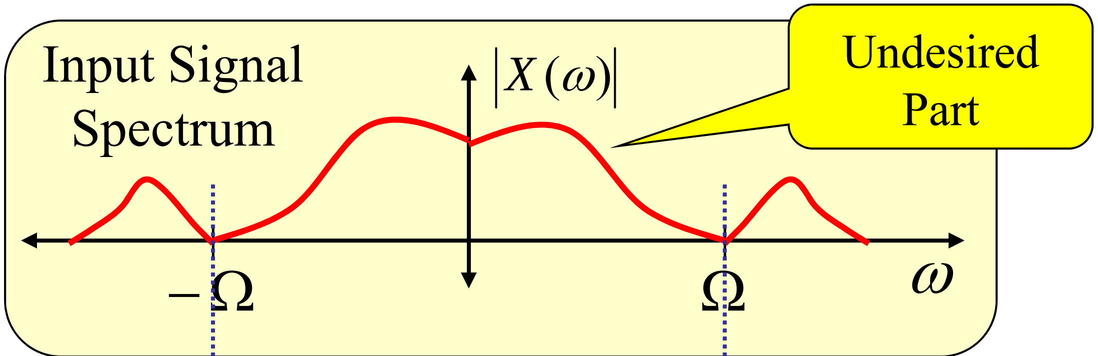
Mathematically:

$$|H(\omega)| = \begin{cases} 1, & -\Omega < \omega < \Omega \\ 0, & otherwise \end{cases}$$

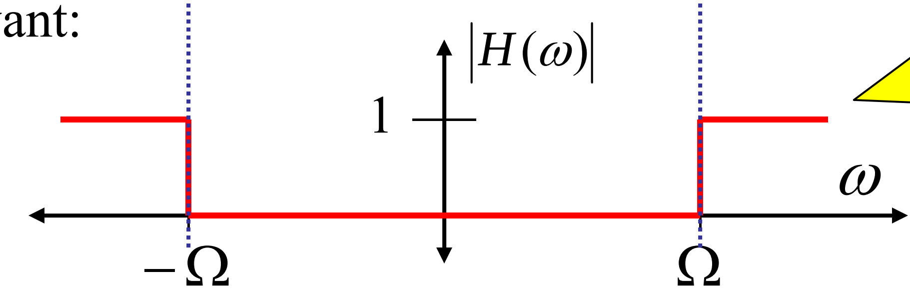
"Passband"

"Stopband"

Case #2:



We then want:



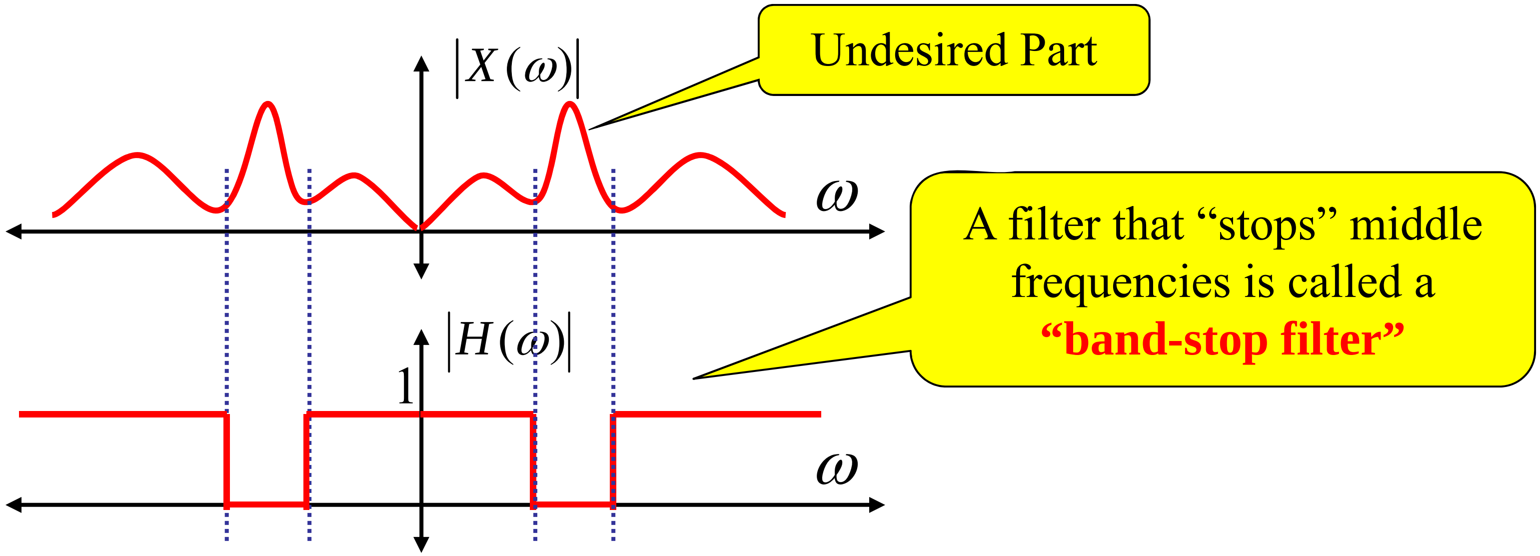
A filter that “passes” high frequencies is called a **“high-pass filter”**

$$|H(\omega)| = \begin{cases} 0, & -\Omega < \omega < \Omega \\ 1, & otherwise \end{cases}$$

“Stopband”

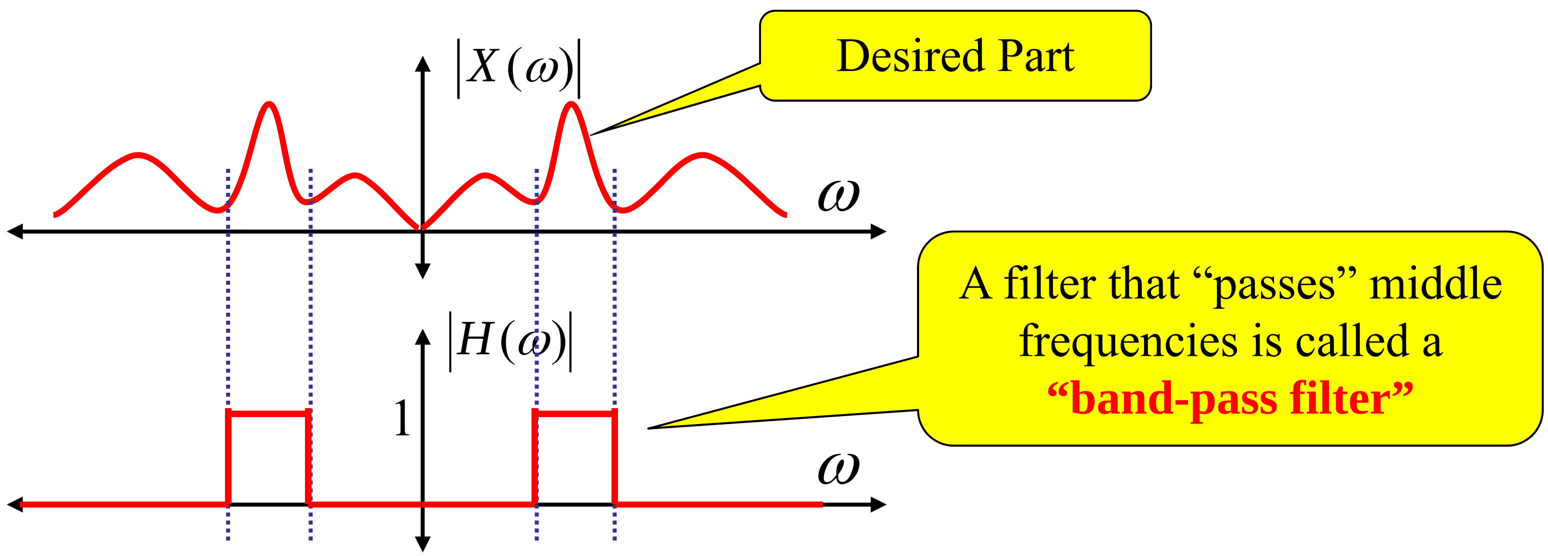
“Passband”

Case # 3:



$|X(\omega)|$
Undesired Part
ω
A filter that "stops" middle frequencies is called a
"band-stop filter"
$|H(\omega)|$
1
ω

Case #4:

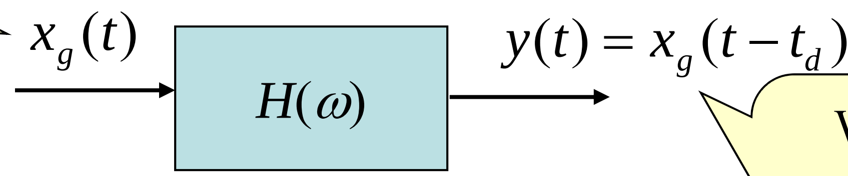


$|X(\omega)|$
Desired Part
ω
A filter that "passes" middle frequencies is called a
"band-pass filter"
$|H(\omega)|$
1
ω

What about the ***phase*** of an IDEAL filter's $H(\omega)$?

Well…we could tolerate a small delay in the output so…

Put in the signal we want "passed"



Want to get out the signal we want "passed"… but we can accept a "small" delay

From the time-shift property of the FT then we need:

$$Y(\omega) = X_g(\omega)e^{-j\omega t_d}$$

Thus we should treat the exponential term here as $H(\omega)$, so we have:

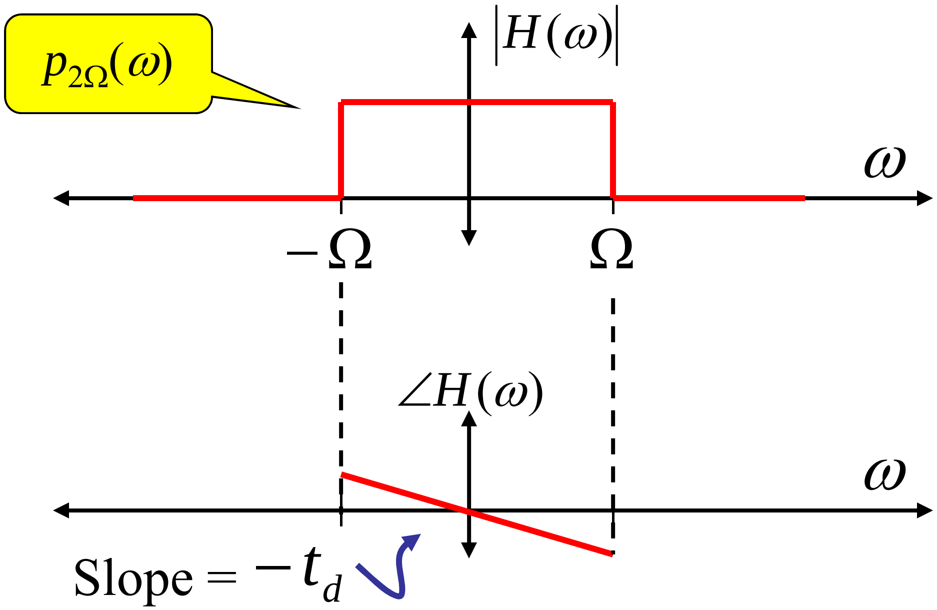
$$|H(\omega)| = \left|e^{-j\omega t_d}\right| = 1$$

For ω in the "pass band" of the filter

$$\angle H(\omega) = \angle e^{-j\omega t_d} = -\omega t_d$$

Line of slope $-t_d$ "Linear Phase"

So… for an ideal low-pass filter (LPF) we have:



$$H(\omega) = \begin{cases} 1e^{-j\omega t_d}, & -\Omega < \omega < \Omega \\ 0, & otherwise \end{cases}$$

$$H(\omega) = p_{2\Omega}(\omega)e^{-j\omega t_d}$$

Phase is undefined in stop band:

$$0 = 0e^{j\theta}$$

$$\angle 0 = ?$$

i.e. phase is undefined for frequencies outside the ideal passband

Summary of *Ideal* Filters

1. Magnitude Response:
   a. Constant in Passband
   b. Zero in Stopband
2. Phase Response
   a. Linear in Passband (negative slope = delay)
   b. Undefined in Stopband

Example of the effect of a nonlinear phase but an ideal magnitude

0 Hz **1 Hz** **2 Hz** **3 Hz**

$$x(t) = 9 - 5\cos(2\pi t) - 3\cos(2\pi 2t) - \cos(2\pi 3t)$$

$H(\omega)$

$$y(t) = 9 - 5\cos(2\pi t - \frac{\pi}{4}) - 3\cos(2\pi 2t - \pi) - \cos(2\pi 3t - \frac{\pi}{10})$$

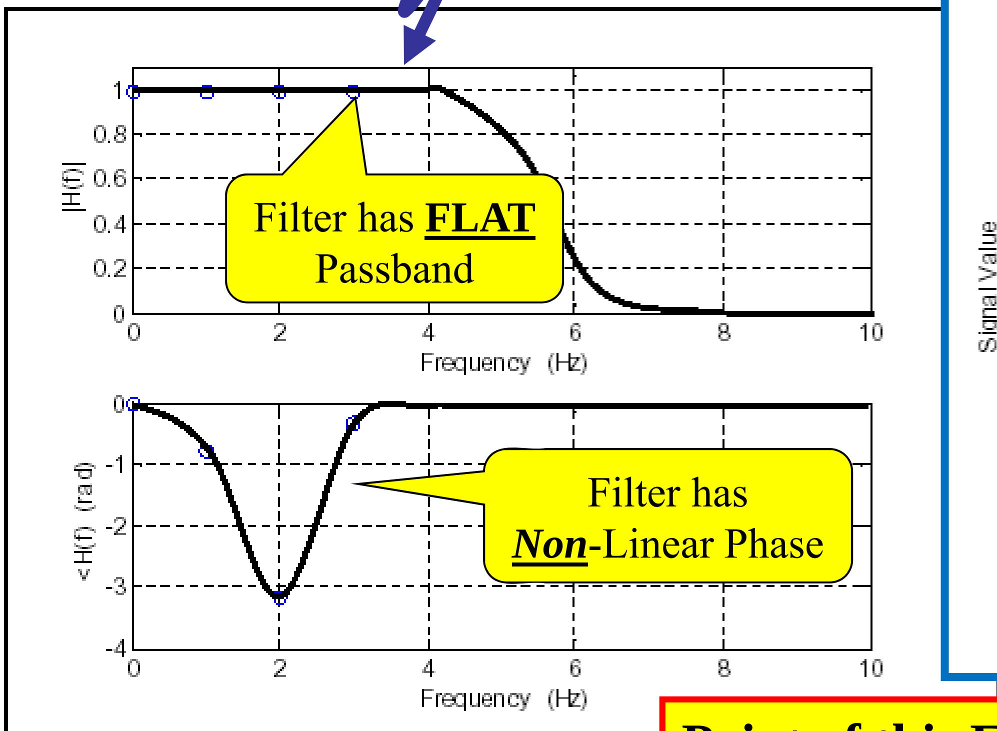
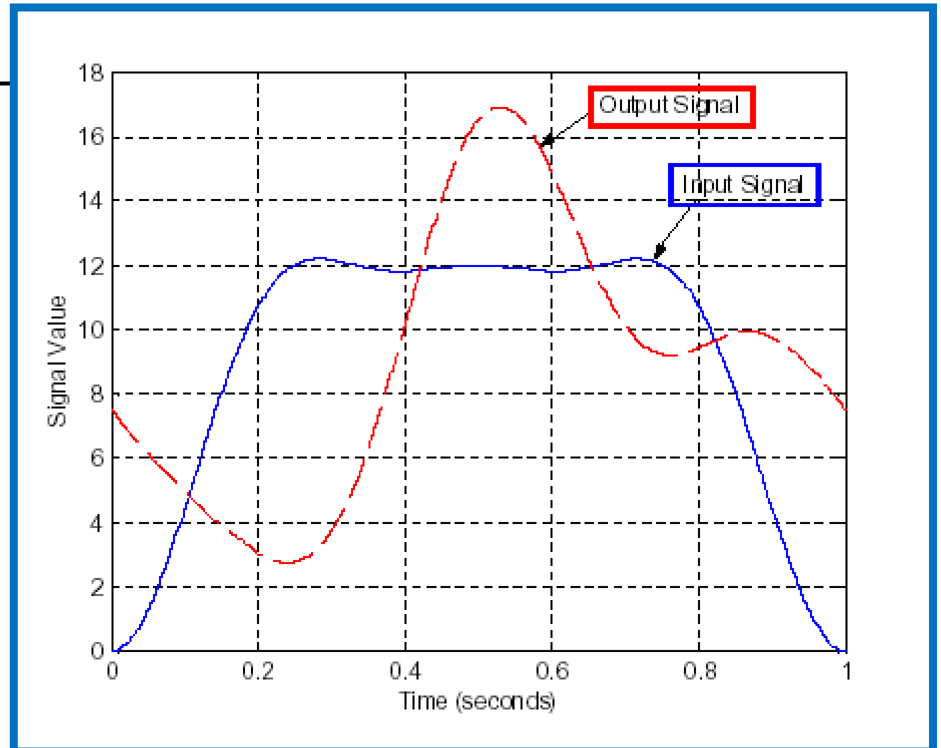


Figure 2: Filter's Frequency Response



Point of this Example

A filter with an ideal magnitude response but non-ideal phase response can still degrade a signal!!!

Are Ideal Filters Realizable? (i.e., can we actually MAKE one?)

Sadly… No!!

So… a big part of CT filter design focuses on how to **get close to the ideal**.

Can't Get an Ideal Filter… Because they are Non-Causal!!!

For the ideal LPF we had $H(\omega) = p_{2\Omega}(\omega)e^{-j\omega t_d}$

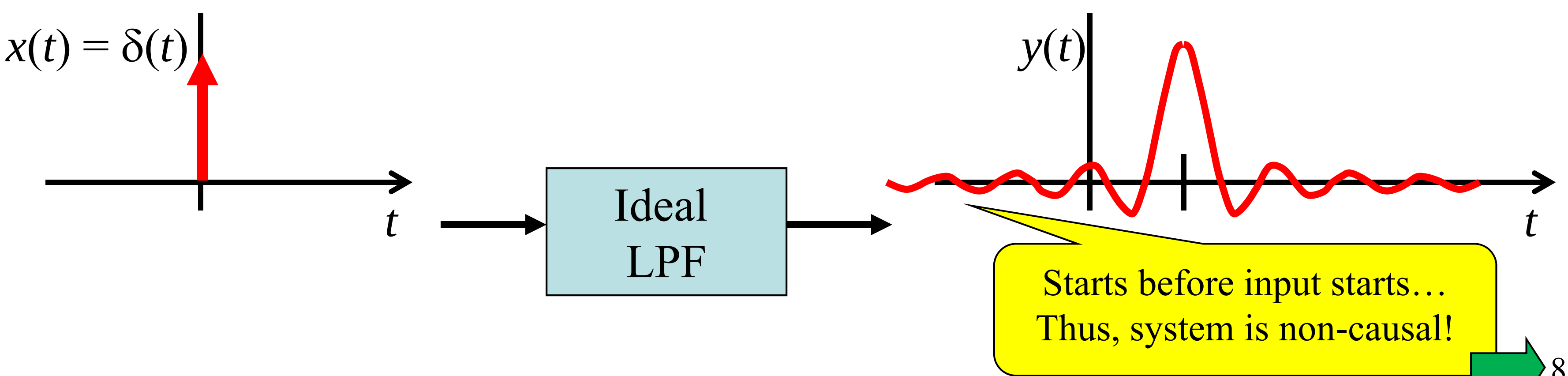
Now consider applying a delta function as its input: $x(t) = \delta(t) \leftrightarrow X(\omega) = 1$

Then the output has FT $Y(\omega) = X(\omega)H(\omega) = p_{2\Omega}(\omega)e^{-j\omega t_d}$

Linear Phase Imparts Delay

From the FT Table: $2\Omega \operatorname{sinc}\left[2\Omega t / 2\pi\right] \longleftrightarrow 2\pi p_{2\Omega}(\omega)$

So the response to a delta (applied at $t = 0$) is: $y(t) = (\Omega / \pi)\operatorname{sinc}\left[(\Omega / \pi)(t - t_d)\right]$



Plotting Frequency Response of Practical Filters

Although we've previously shown the plots of Freq. Resp. using the actual numerical values of $|H(\omega)|$ it is VERY common to plot its ***decibel*** values.

Decibel: a **logarithmic unit** of measure for a **ratio** between **two powers**

$$\underbrace{10\underbrace{\log_{10}\left(\frac{P_1}{P_2}\right)}_{\text{bel}}}_{\text{decibel}}$$

P_1/P_2 (non-dB)	P_1/P_2 (dB)
$1000 = 10^3$	30 dB
$100 = 10^2$	20 dB
$10 = 10^1$	10 dB
$1 = 10^0$	0 dB
$0.1 = 10^{-1}$	–10 dB
$0.01 = 10^{-2}$	–20 dB
$0.001 = 10^{-3}$	–30 dB

Know These!

Decibel Power Rules

30 dB is "P ratio of 1000"

20 dB is "P ratio of 100"

10 dB is "P ratio of 10"

0 dB is "P ratio of 1"

-10 dB is "P ratio of 0.1"

-20 dB is "P ratio of 0.01"

-30 dB is "P ratio of 0.001"

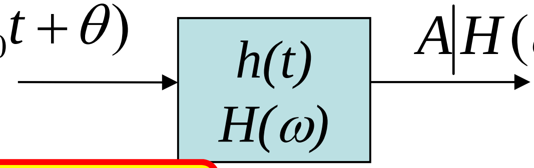
Another "Rule" to Know!!

$P_1/P_2 = 2$ ➔ ~ 3 dB

$P_1/P_2 = 1/2$ ➔ ~ -3 dB

But… $|H(\omega)|$ relates Voltages (or current)… not POWER!!!

$A\cos(\omega_0 t + \theta)$



$A|H(\omega_0)|\cos(\omega_0 t + \theta + \angle H(\omega_0))$

Input voltage amplitude $= A$

Output voltage amplitude $= A|H(\omega_0)|$

$$|H(\omega_0)| = \frac{\text{Output Voltage Amplitude}}{\text{Input Voltage Amplitude}}$$

Convert to Powers

Input Power = $A^2/2$

Output Power = $A^2|H(\omega_0)|^2/2$

$$10\log_{10}\left(\frac{P_{out}}{P_{in}}\right) = 10\log_{10}\left(\frac{A^2|H(\omega)|^2/2}{A^2/2}\right)$$

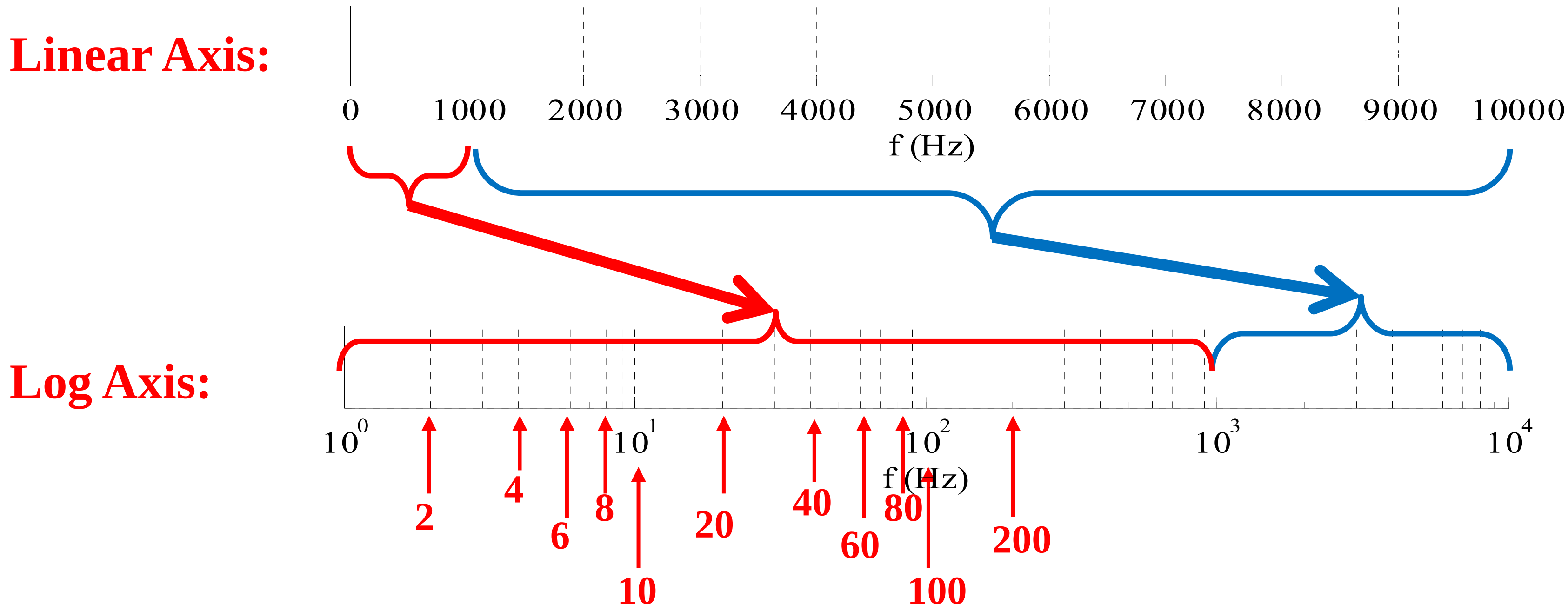
$$= 10\log_{10}\left(|H(\omega)|^2\right)$$

$$= 20\log_{10}\left(|H(\omega)|\right)$$

$$\mathbf{20\log_{10}(|H(\omega)|)}$$

Decibel value for $|H(\omega_0)|$

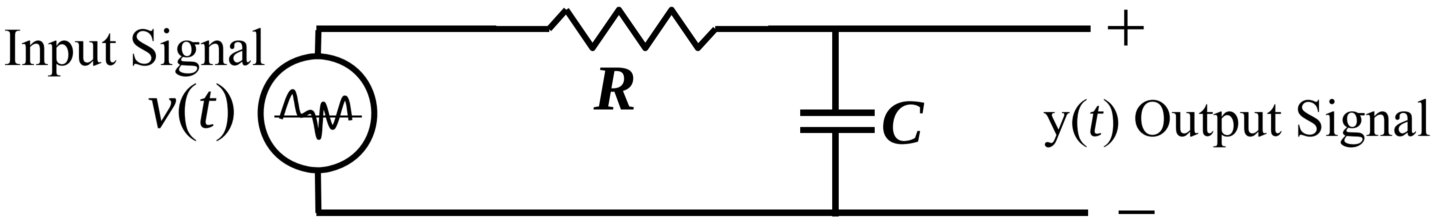
In addition to using decibels for the $|H(\omega)|$ it is also common to use a **logarithmic scale** for the frequency axis



We may be just as interested in 0 – 1 kHz as we are in 1 – 10 kHz

- But the linear axis plot has the 0 – 1 kHz region all "scrunched up"
- However… the log axis allows us to expand out the lower frequencies to see them better!

Simplest Real-World Lowpass Filter: RC Circuit

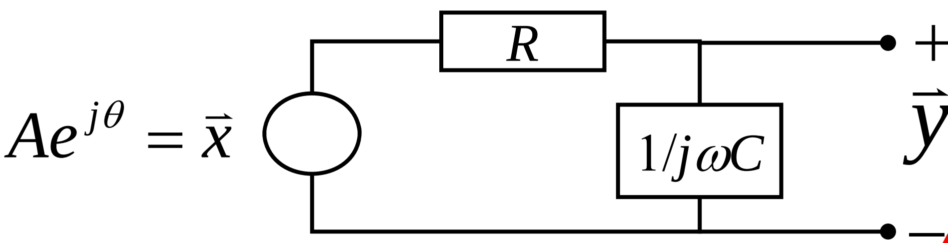


1. Convert capacitor into impedance: $Z_c(\omega) = \dfrac{1}{j\omega C}$

Small impedance at high ω
Large impedance at low ω

2. Imagine input as phasor: $Ae^{j\theta} = \vec{x}$

3. Now analyze the circuit as if it were a DC circuit with a complex voltage in (the phasor) and complex resistors (the impedances):



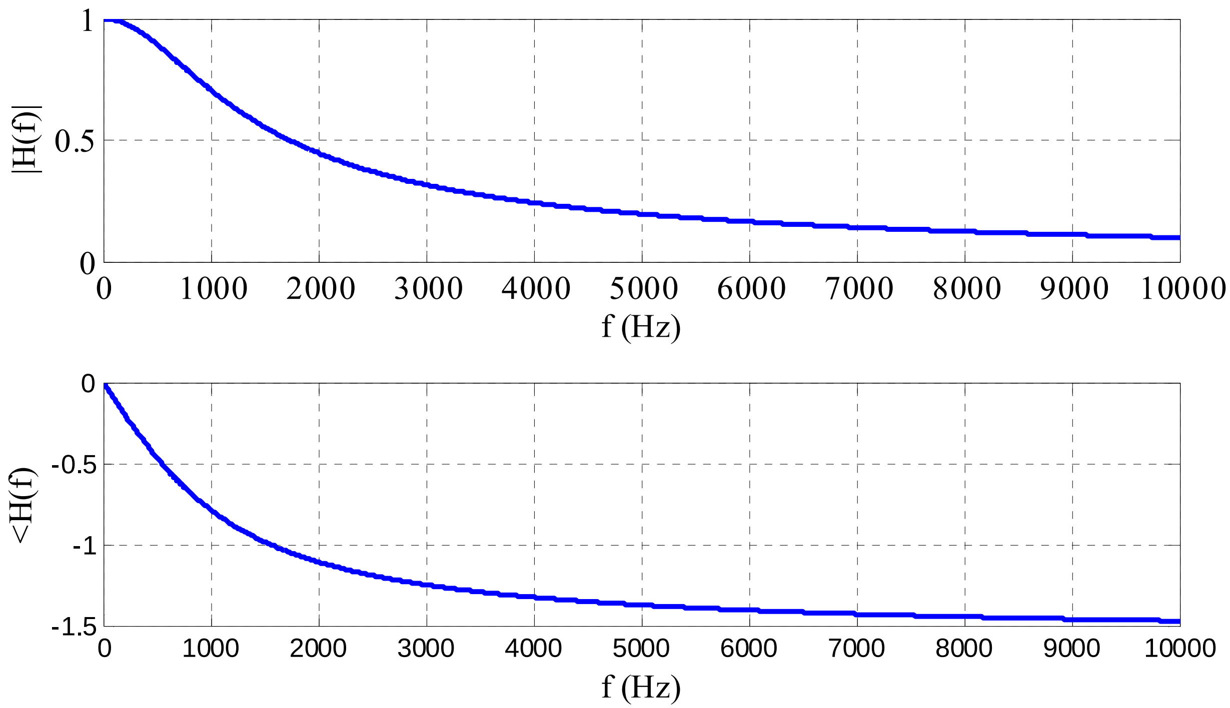
Now find the output phasor as a function of the input phasor… the thing that multiplies the input phasor is ALWAYS the Freq Resp !

Here… use Voltage Divider.

$$\vec{y} = \frac{Z_c(\omega)}{R + Z_c(\omega)}\vec{x} = \left[\frac{1}{1 + j\omega RC}\right]\vec{x}$$

$$H(\omega) = \left[\frac{1}{1 + j\omega RC}\right]$$

Now… we can plot this

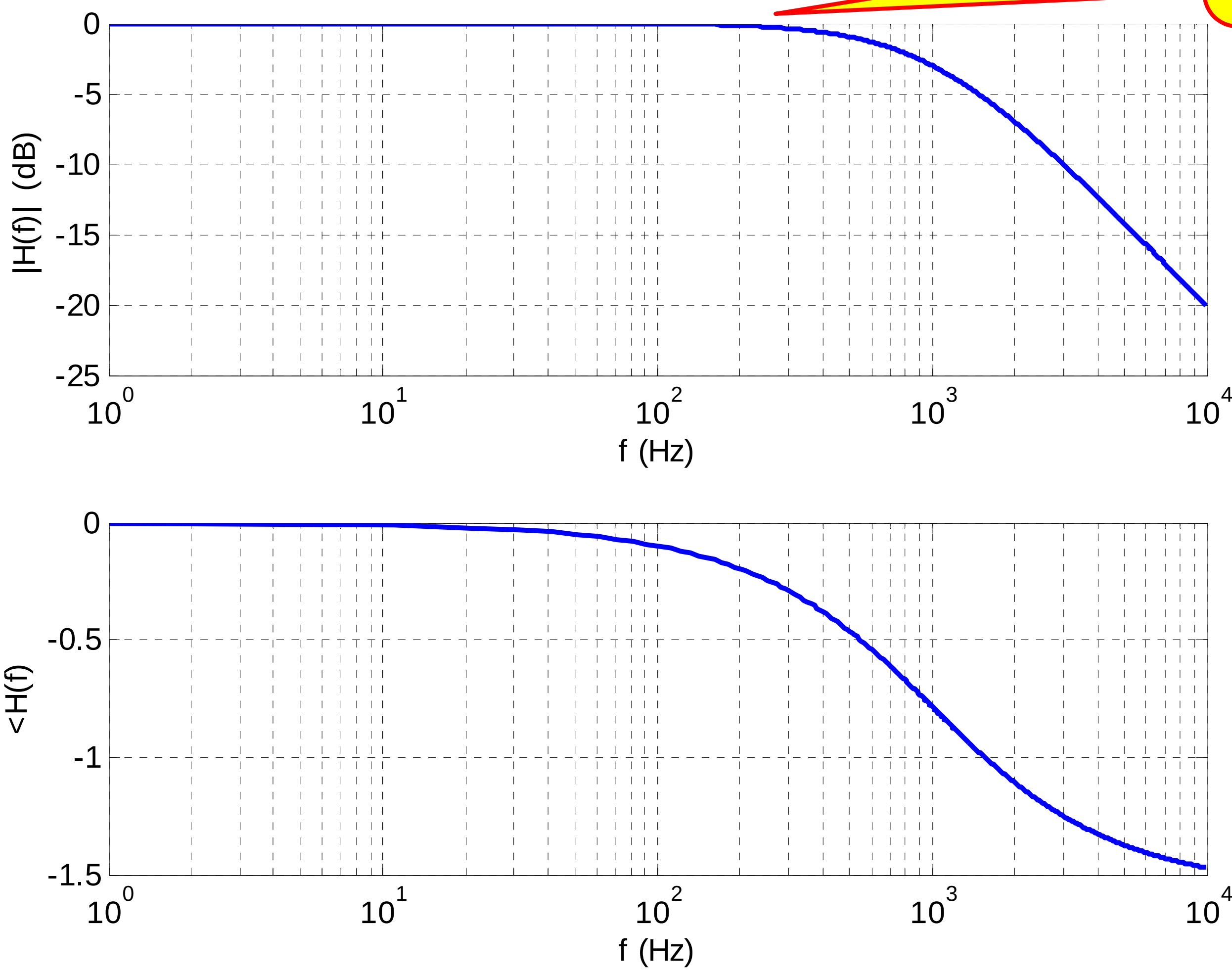



```
RC=1.5915e-4;
f=0:10:10000;
H=1./(1 + j*2*pi*f*RC);
subplot(2,1,1)
plot(f,abs(H))
grid
xlabel('f (Hz)')
ylabel('|H(f)|')
subplot(2,1,2)
plot(f,angle(H))
grid
xlabel('f (Hz)')
ylabel('<H(f)')
```

Although these are "correct" plots… we usually prefer to use:

- dB for the magnitude axis (but not the angle axis!)
- log axis (rather than linear) for the frequency axis
  - ***But… keep in mind that when using a log axis a linear phase will NOT be a straight line!!!***

Instead… we can plot this using dB and log axes…



Using log scale allows us to see that this filter is “quite flat” up to about 200 Hz!

```
RC=1.5915e-4;
f=1:10:10000;
H=1./(1 + j*2*pi*f*RC);
subplot(2,1,1)
semilogx(f,20*log10(abs(H)))
grid
xlabel('f (Hz)')
ylabel('|H(f)|')
subplot(2,1,2)
semilogx(f,angle(H))
grid
xlabel('f (Hz)')
ylabel('<H(f)')
```

Use “20” here!