BINGHAMTON UNIVERSITY

State University of New York

EECE 301
Signals & Systems
Prof. Mark Fowler

<u>Note Set #13</u>

- C-T Signals: Fourier Transform Properties

Fourier Transform Properties

These properties are useful for two main things:

1. They help you apply the table to a wider class of signals
2. They are often the key to understanding how the FT can be used in a given application.

So… even though these results may at first seem like "just boring math" they are important tools that let signal processing engineers understand how to build things like cell phones, radars, mp3 processing, etc.

Here… we will only cover the most important properties.

See the available table for the complete list of properties!

In this note set we simply learn these most-important properties… in the next note set we'll see how to use them.

1. Linearity (Supremely Important)

Gets used virtually all the time!!

If $x(t) \leftrightarrow X(\omega)$ & $y(t) \leftrightarrow Y(\omega)$

then $[ax(t) + by(t)] \leftrightarrow [aX(\omega) + bY(\omega)]$

Another way to write this property:

$$\mathfrak{F}\{ax(t) + by(t)\} = a\mathfrak{F}\{x(t)\} + b\mathfrak{F}\{y(t)\}$$

To see why: $\mathfrak{F}\{ax(t) + by(t)\} = \int_{-\infty}^{\infty} [ax(t) + by(t)] e^{-j\omega t} dt$

Use Defn of FT

$$= a\underbrace{\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt}_{= X(\omega)} + b\underbrace{\int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt}_{= Y(\omega)}$$

By standard Property of Integral of sum of functions

By Defn of FT

2. Time Shift (Really Important!)

Used often to understand practical issues that arise in audio, communications, radar, etc.

$$\text{If} \quad x(t) \leftrightarrow X(\omega) \quad \text{then} \quad x(t-c) \leftrightarrow X(\omega)e^{-jc\omega}$$

Note: If $c > 0$ then $x(t-c)$ is a **delay** of $x(t)$

So… what does this *mean*??

First… it does nothing to the magnitude of the FT: $\left|X(\omega)e^{-j\omega c}\right| = \left|X(\omega)\right|$

That means that a shift doesn't change "how much" we need of each of the sinusoids we build with

Second… it does change the phase of the FT: $\angle\{X(\omega)e^{-jc\omega}\} = \angle X(\omega) + \angle e^{-jc\omega}$

$$= \angle X(\omega) + c\omega$$

Line of slope –*c*

Phase shift increases linearly as the frequency increases

This gets added to original phase

Shift of Time Signal ⇔ "Linear" Phase Shift of Frequency Components

3. Time Scaling (Important)

Q: If $x(t) \leftrightarrow X(\omega)$, then $x(at) \leftrightarrow ???$ for $a \neq 0$

A: $$x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

If the time signal is Time Scaled by *a*

Then… The FT is Freq. Scaled by 1/a

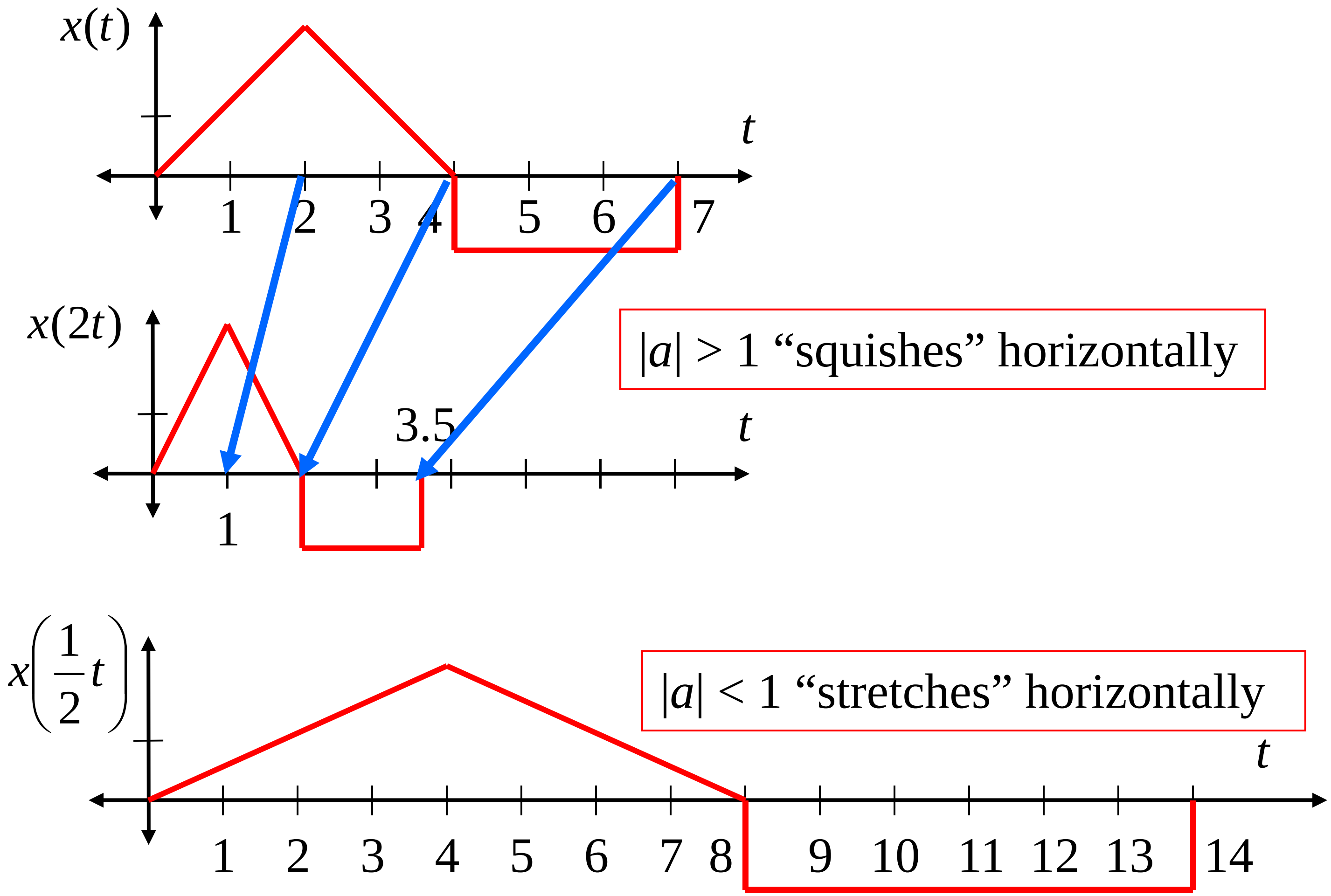
An interesting "duality"!!!

To explore this FT property…first, what does $x(at)$ look like?

Original Signal

Time-Scaled w/ $a = 2$

Time-Scaled w/ $a = 1/2$



$|a| > 1$ makes it "wiggle" faster $\Rightarrow$ need <u>more</u> high frequencies

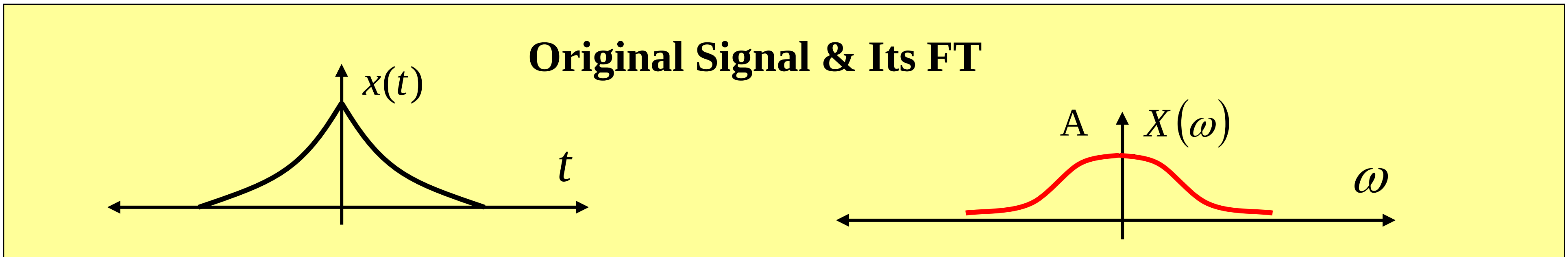
$|a| < 1$ makes it "wiggle" slower $\Rightarrow$ need <u>less</u> high frequencies

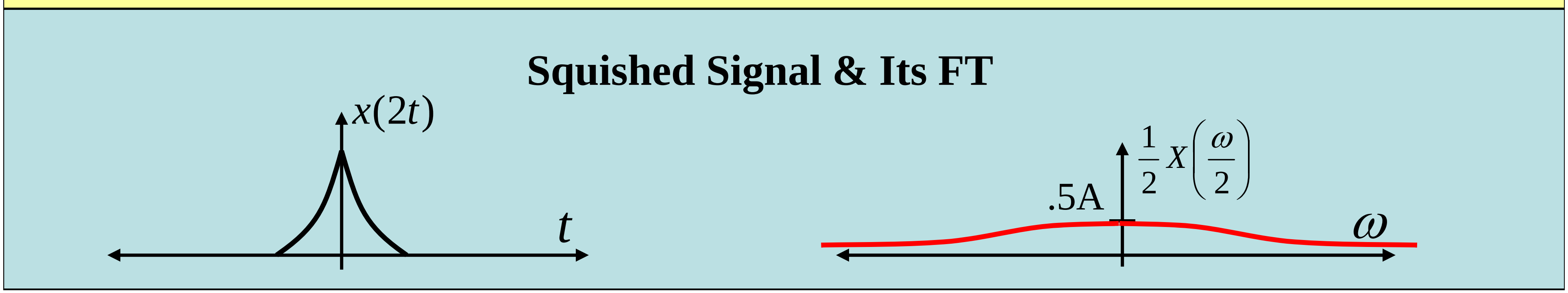
<u>When |*a*| > 1 $\Rightarrow$ |1/*a*| < 1</u>

$$x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

Time Signal is Squished

FT is Stretched Horizontally and Reduced Vertically



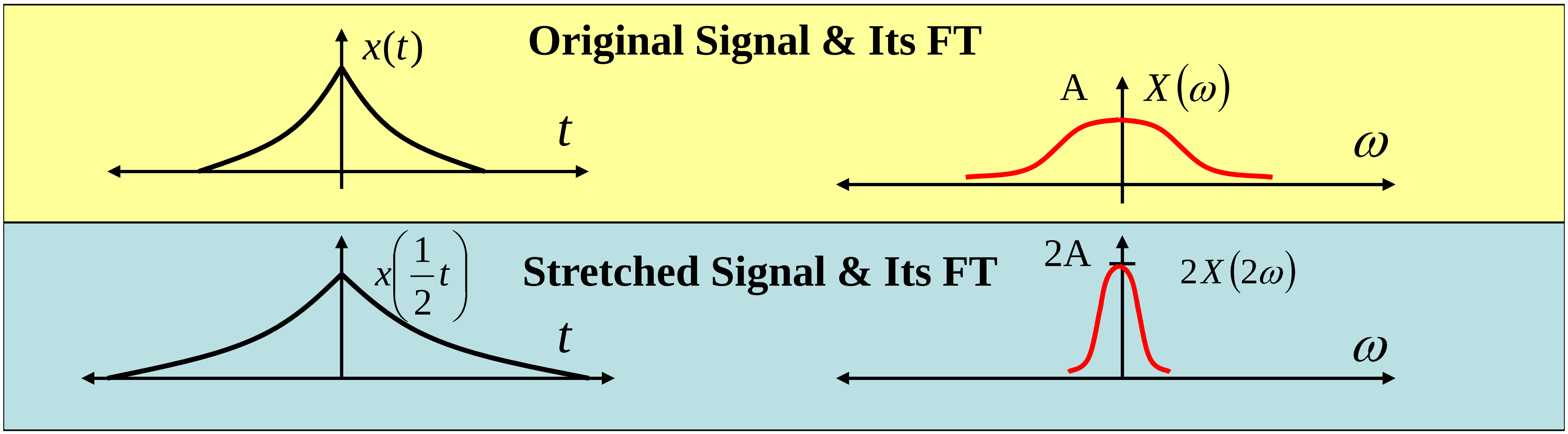


<u>When $|a| < 1 \;\Rightarrow\; |1/a| > 1$</u>

$$x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

Time Signal is Stretched

FT is Squished Horizontally
and Increased Vertically



<u>Rough</u> Rule of Thumb we can extract from this property:

$\uparrow$ Duration $\Rightarrow$ $\downarrow$ Bandwidth

$\downarrow$ Duration $\Rightarrow$ $\uparrow$ Bandwidth

Very Short Signals *tend* to take up Wide Bandwidth

4. Time Reversal (Special case of time scaling: $a = -1$)

$$x(-t) \leftrightarrow X(-\omega)$$

Note: $X(-\omega) = \int_{-\infty}^{\infty} x(t)e^{-j(-\omega)t}dt \;=\; \overline{\overline{\int_{-\infty}^{\infty} x(t)e^{+j\omega t}dt}}$

double conjugate = "No Change"

$$= \overline{\int_{-\infty}^{\infty} \overline{x(t)}\,\overline{e^{+j\omega t}}dt}$$

Conjugate changes to $-j$

$\overline{x(t)}$ = $x(t)$ if $x(t)$ is real

$$= \overline{\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt} = \overline{X(\omega)}$$

Recall: conjugation doesn't change abs. value but negates the angle

$$\left|\overline{X(\omega)}\right| = \left|X(\omega)\right|$$

$$\angle\overline{X(\omega)} = -\angle X(\omega)$$

So if $x(t)$ is real, then we get the special case:

$$x(-t) \leftrightarrow \overline{X(\omega)}$$

5. Modulation Property

Super important!!!

Essential for understanding <u>practical</u> issues that arise in <u>communications</u>, <u>radar</u>, etc.

There are two forms of the modulation property…

1. **Complex Exponential Modulation** … simpler mathematics, doesn't *<u>directly</u>* describe real-world cases
2. **Real Sinusoid Modulation**… mathematics a bit more complicated, directly describes real-world cases

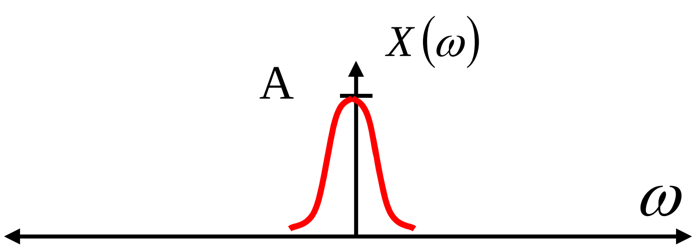
Euler's formula connects the two… so you often can use the Complex Exponential form to analyze real-world cases

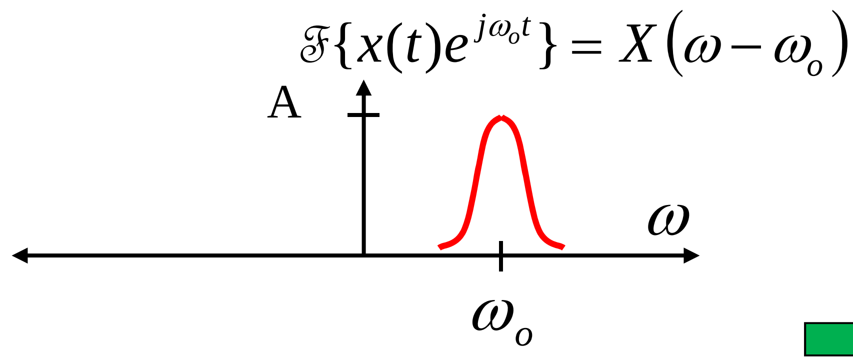
<u>Complex Exponential Modulation Property:</u>

$$x(t)e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0)$$

Multiply signal by a complex sinusoid

Shift the FT in frequency





Real Sinusoid Modulation

Based on Euler, Linearity property, & the Complex Exp. Modulation Property

$$\mathfrak{F}\{x(t)\cos(\omega_0 t)\} = \mathfrak{F}\left\{\frac{1}{2}\left[x(t)e^{j\omega_0 t} + x(t)e^{-j\omega_0 t}\right]\right\}$$

Euler's Formula

$$= \frac{1}{2}\left[\mathfrak{F}\left\{\left[x(t)e^{j\omega_0 t}\right]\right\} + \mathfrak{F}\left\{\left[x(t)e^{-j\omega_0 t}\right]\right\}\right]$$

Linearity of FT

$$= \frac{1}{2}\left[X(\omega - \omega_o) + X(\omega + \omega_o)\right]$$

Comp. Exp. Mod.

The Result:

$$x(t)\cos(\omega_0 t) \leftrightarrow \frac{1}{2}\left[X(\omega+\omega_0) + X(\omega-\omega_0)\right]$$

Shift Down Shift Up

Related Result:

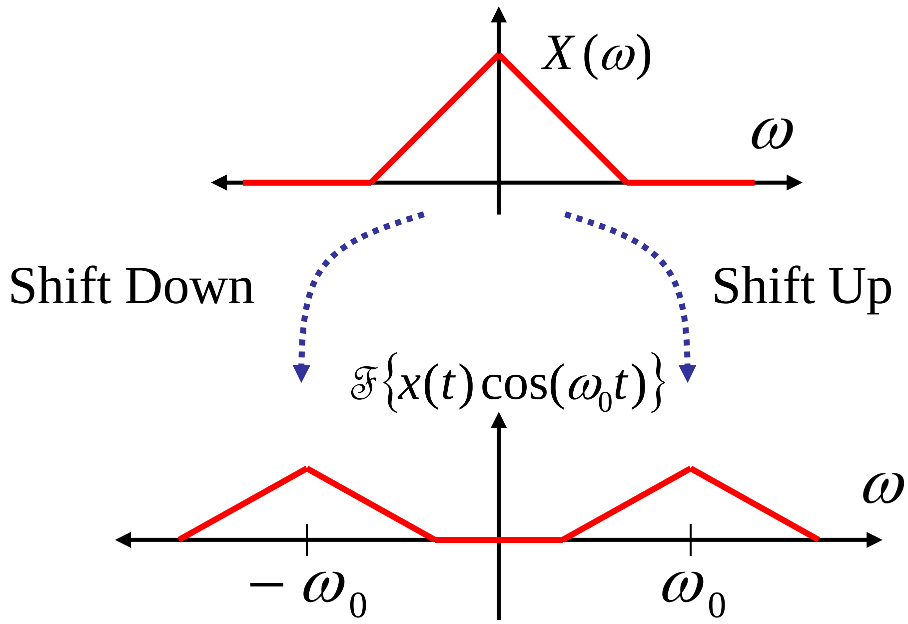
$$x(t)\sin(\omega_0 t) \leftrightarrow \frac{j}{2}\left[X(\omega+\omega_0) - X(\omega-\omega_0)\right]$$

Exercise: $x(t)\cos(\omega_0 t + \phi_0) \leftrightarrow ??$

Visualizing the Result

$$x(t)\cos(\omega_0 t) \leftrightarrow \frac{1}{2}\left[X(\omega-\omega_0)+X(\omega+\omega_0)\right]$$

Shift up   Shift down



Interesting… This tells us how to move a signal's spectrum up to higher frequencies without changing the shape of the spectrum!!!

What is that good for??? Well… only high frequencies will radiate from an antenna and propagate as electromagnetic waves and then induce a signal in a receiving antenna….

6. Convolution Property (The Most Important FT Property!!!)

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \quad \leftrightarrow \quad Y(\omega) = X(\omega)H(\omega)$$

We will not yet discuss the "*Convolution*" aspect of this now… but we will talk about it in depth later.

In the next Note Set we will explore the real-world use of the right side of this result!

7. Parseval's Theorem (Recall Parseval's Theorem for FS!)

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

Energy computed in time domain

Energy computed in frequency domain

$$|x(t)|^2 dt$$

= energy at time t

$$|X(\omega)|^2 \frac{d\omega}{2\pi}$$

= energy at freq. ω

8. Duality:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$x(t) \qquad\qquad X(\omega)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

Both FT & IFT are pretty much the "same machine": $c\int_{-\infty}^{\infty} f(\lambda)e^{\pm j\lambda\xi}d\lambda$

So if there is a "time-to-frequency" property we would expect a virtually similar "frequency-to-time" property

Illustration: Delay Property: $x(t-c) \leftrightarrow X(\omega)e^{-j\omega c}$

Modulation Property: $x(t)e^{j\omega_0 t} \leftrightarrow X(\omega-\omega_0)$

Other Dual Properties: (Multiply by t^n) vs. (Diff. in time domain)

(Convolution) vs. (Mult. of signals)

Also, this duality structure gives FT pairs that show duality.

Suppose we have a FT table that a FT Pair A… we can get the dual Pair B using the general Duality Property:

1. Take the FT side of (known) Pair A and replace ω by *t* and move it to the time-domain side of the table of the (unknown) Pair B.
2. Take the time-domain side of the (known) Pair A and replace *t* by –ω, multiply by 2π, and then move it to the FT side of the table of the (unknown) Pair B.

Here is an example… We found the FT pair for the pulse signal: