BINGHAMTON UNIVERSITY
State University of New York

EECE 301
Signals & Systems
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Note Set #11

- C-T Signals: Fourier Transform Concept (for Non-Periodic Signals)

Intro to Fourier Transform

Recall: Fourier Series represents a periodic signal as a sum of sinusoids

or complex sinusoids $e^{jk\omega_0 t}$

Note: Because the FS uses "harmonically related" frequencies $k\omega_0$, it can only create periodic signals

Q: Can we modify the FS idea to handle non-periodic signals?

A: Yes!!

What about $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\omega_k t}$?

With *arbitrary* discrete frequencies… NOT harmonically related

This will give some non-periodic signals but not all signals of interest!!

The problem with this is that it cannot include all possible frequencies!

No matter how close we try to choose the discrete frequencies ω_k there are always some left out of the sum!!!

We need some way to include ALL frequencies!!

How about:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$$

Yes… this will work for any *practical* non-periodic signal!!

Called the "Fourier Integral" also, more commonly, called the "**Inverse Fourier Transform**"

Plays the role of c_k

Plays the role of $e^{jk\omega_0 t}$

Integral replaces sum because it can "add up over the ***continuum*** of frequencies"!

Okay… given $x(t)$ how do we get $X(\omega)$?

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

Called the "**Fourier Transform**" of $x(t)$

Note: $X(\omega)$ is complex-valued function of $\omega \in (-\infty, \infty)$

$|X(\omega)|$ $\quad$ $\angle X(\omega)$

Need to use two plots to show it

Comparison of FT and FS

Fourier Series: Used for periodic signals

Fourier Transform: Used for non-periodic signals (although we will see later that it can also be used for periodic signals)

	Synthesis	**Analysis**
Fourier Series	$x(t)=\sum_{n=-\infty}^{\infty} c_k e^{jk\omega_0 t}$ **Fourier Series**	$c_k = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-jk\omega_0 t}dt$ **Fourier Coefficients**
Fourier Transform	$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$ **Inverse Fourier Transform**	$X(\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$ **Fourier Transform**

FS coefficients c_k are a complex-valued function of integer k

FT $X(\omega)$ is a complex-valued function of the variable $\omega \in (-\infty, \infty)$

Synthesis Viewpoints:

FS: $$x(t) = \sum_{n=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

$|c_k|$ shows how much there is of the signal at frequency $k\omega_0$

$\angle c_k$ shows how much phase shift is needed at frequency $k\omega_0$

We need two plots to show these

FT: $$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$$

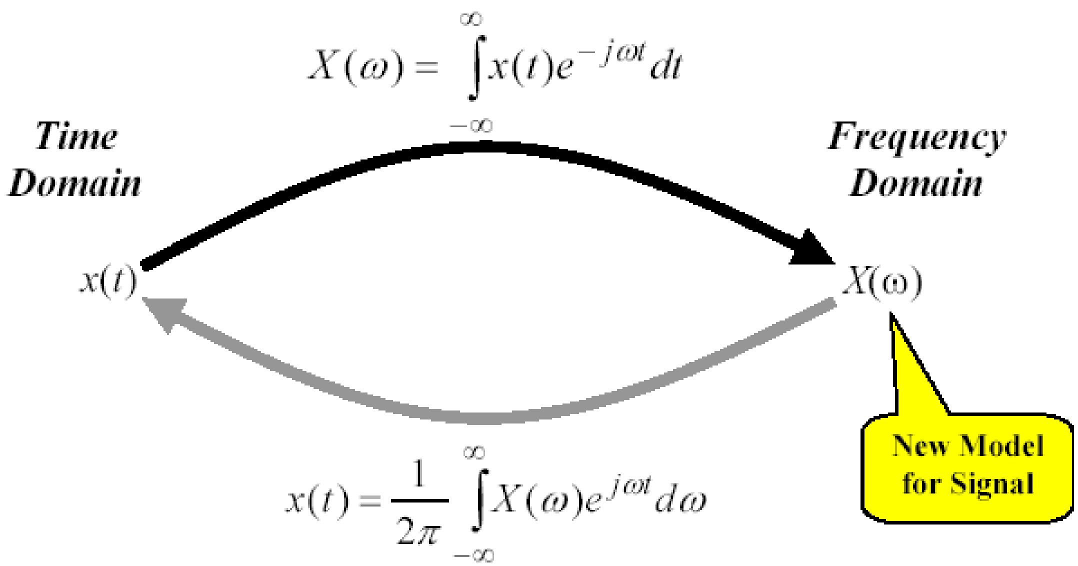
$|X(\omega)|$ shows how much there is in the signal at frequency ω

$\angle X(\omega)$ shows how much phase shift is needed at frequency ω

We need two plots to show these

Fourier Transform Viewpoint

View FT as a transformation into a new "domain"



$x(t)$ is the "time domain" description of the signal
$X(\omega)$ is the "frequency domain" description of the signal

Alternate Notations

1. $x(t) \leftrightarrow X(\omega)$
2. $X(\omega) = \mathfrak{F}\{x(t)\}$

   $\Rightarrow$ $\mathfrak{F}\{\ \}$ is an "operator on" $x(t)$ to give $X(\omega)$
3. $x(t) = \mathfrak{F}^{-1}\{X(\omega)\}$

   $\Rightarrow$ $\mathfrak{F}^{-1}\{\}$ is an "operator on" $X(\omega)$ to give $x(t)$

Analogy: Looking at $X(\omega)$ is "like" looking at an x-ray of the signal- in the sense that an x-ray lets you see what is inside the object… shows what stuff it is made from.

In this sense: $X(\omega)$ shows what is "inside" the signal – it shows how much of each complex sinusoid is "inside" the signal

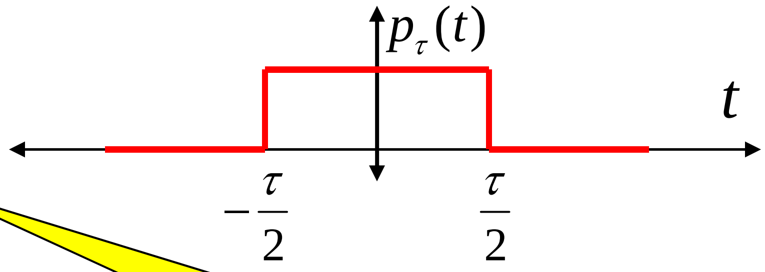
Note: $x(t)$ completely determines $X(\omega)$

$X(\omega)$ completely determines $x(t)$

There are some advanced mathematical issues that can be hurled at these comments… we'll not worry about them

Example: FT of a Rectangular pulse

τ = pulse width



Given: a rectangular pulse signal $p_\tau(t)$

Recall: we use this symbol to indicate a rectangular pulse with width τ

Find: $P_\tau(\omega)$… the FT of $p_\tau(t)$

Note the Notational Convention: lower-case for time signal and corresponding upper-case for its FT

Solution: (Here we'll directly do the integral… but later we'll use the "FT Table")

Note that

$$p_\tau(t) = \begin{cases} 1, & -\frac{\tau}{2} \le t \le \frac{\tau}{2} \\ 0, & otherwise \end{cases}$$

Now apply the definition of the FT:

$$P_\tau(\omega) = \int_{-\infty}^{\infty} p_\tau(t) e^{-j\omega t} dt = \int_{-\tau/2}^{\tau/2} e^{-j\omega t} dt$$

Limit integral to where $p_\tau(t)$ is non-zero… and use the fact that it is 1 over that region

$$= \frac{-1}{j\omega}\left[e^{-j\omega t}\right]_{\frac{-\tau}{2}}^{\frac{\tau}{2}} = \frac{2}{\omega}\left[\frac{e^{j\frac{\omega\tau}{2}} - e^{-j\frac{\omega\tau}{2}}}{j2}\right]$$

Artificially inserted 2 in numerator and denominator

$$= \sin\left(\frac{\omega\tau}{2}\right)$$

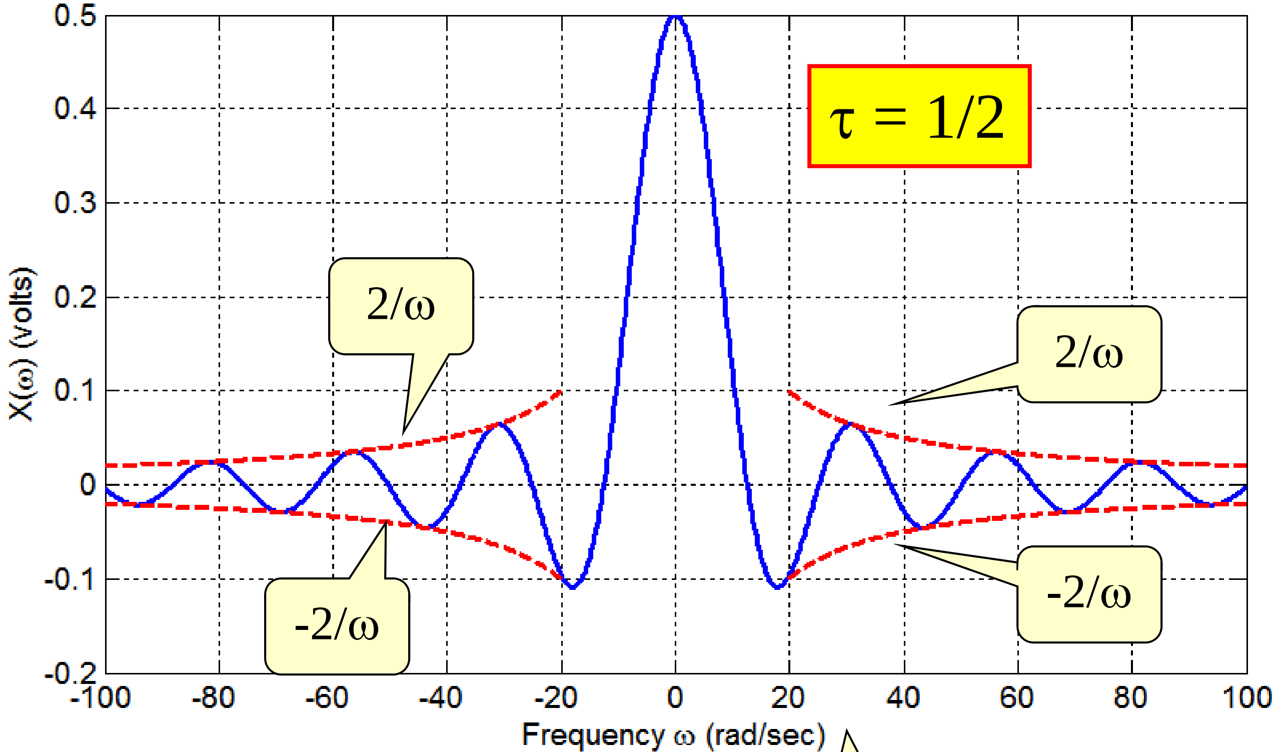
Use Euler's Formula

$$P_\tau(\omega) = \frac{2\sin\left(\frac{\omega\tau}{2}\right)}{\omega}$$

sin goes up and down between -1 and 1

1/ω decays down as |ω| gets big… this causes the overall function to decay down

For ***this*** case the FT is real valued so we can plot it using a single plot (shown in solid blue here):



$$P_\tau(\omega) = \frac{2\sin\left(\frac{\omega\tau}{2}\right)}{\omega}$$

The sine wiggles up & down "between ±2/ω"

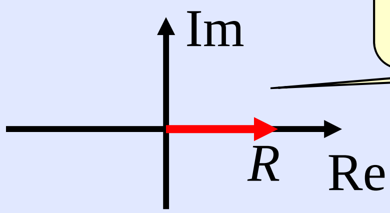
Now… let's think about how to make a magnitude/phase plot…

Even though this FT is real-valued we can still plot it using magnitude and phase plots:

We can view any real number as a complex number that has zero as its imaginary part

A *positive* real number R will have:

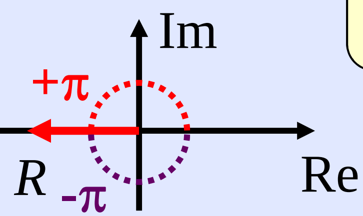
$|R| = R \qquad \angle R = 0$



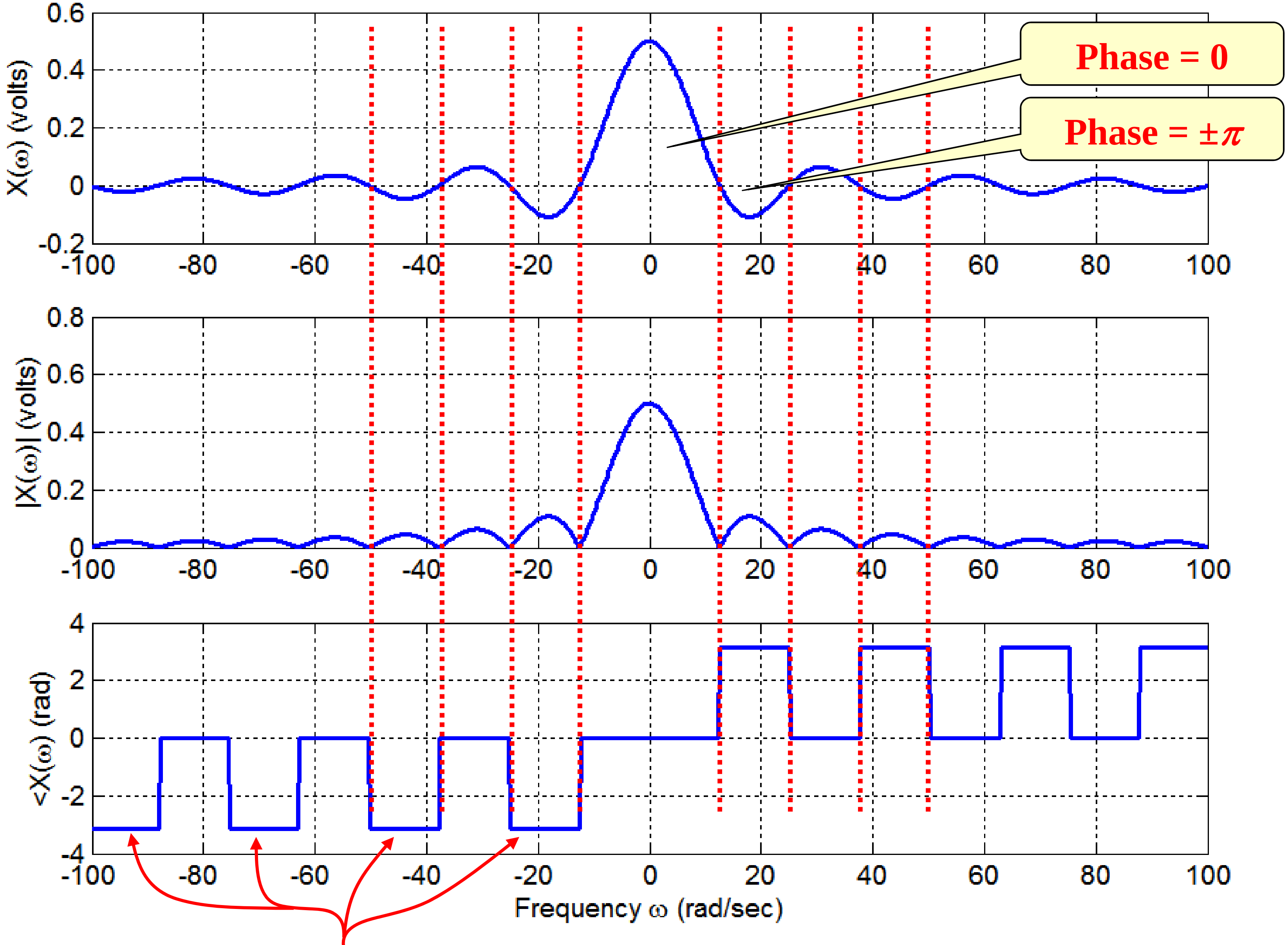
A *negative* real number R will have:

$|R| = -R \qquad \angle R = \pm\pi$

Can use either one!!



Applying these Ideas to the Real-valued FT $P_\tau(\omega)$



Here I have chosen $-\pi$ to display odd symmetry

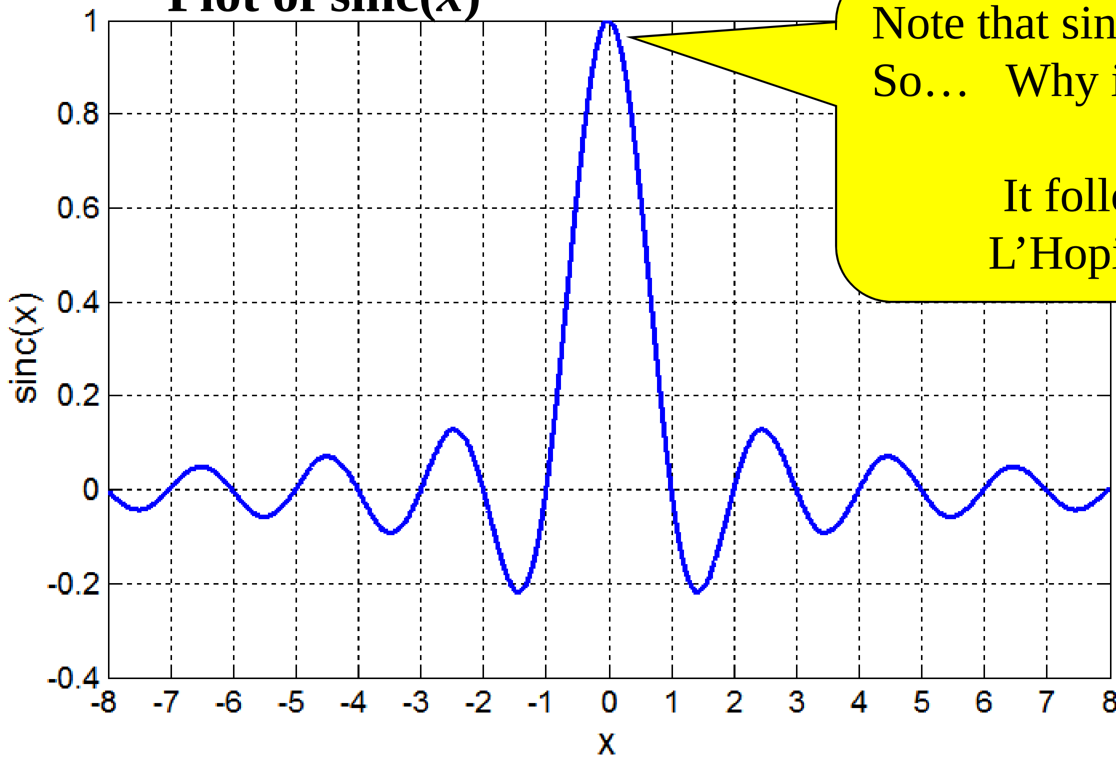
Definition of "Sinc" Function

The result we just found had this mathematical form: $P_\tau(\omega) = \dfrac{2\sin\left(\dfrac{\omega\tau}{2}\right)}{\omega}$

This structure shows up enough that we define a special function to capture it:

Define: $$\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

Plot of sinc(*x*)



Note that sinc(0) = 0/0.
So… Why is sinc(0) = 1?

It follows from L'Hopital's Rule

With a little manipulation we can re-write the FT result for a pulse in terms of the sinc function:

Recall:

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

Need π times something…

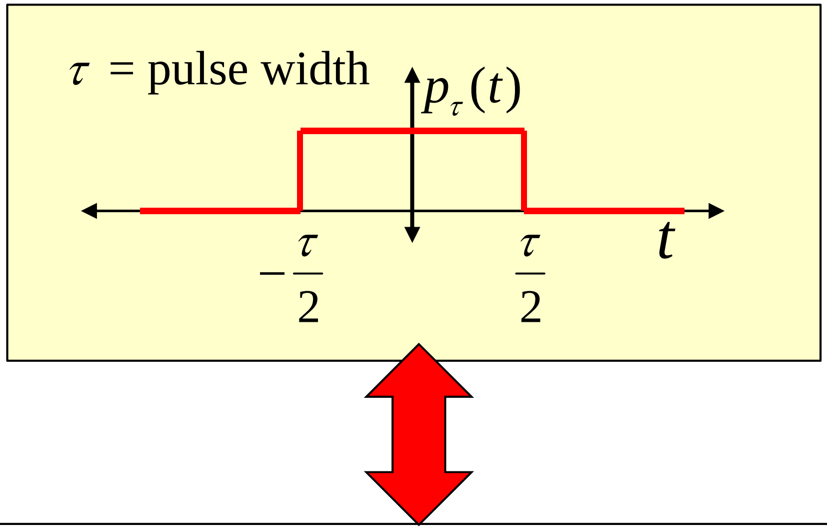
$$P_\tau(\omega) = \frac{2\sin\left(\frac{\omega\tau}{2}\right)}{\omega} = \frac{2\sin\left(\frac{\pi}{\pi}\frac{\omega\tau}{2}\right)}{\omega} = \frac{2\sin\left(\pi\frac{\omega\tau}{2\pi}\right)}{\omega}$$

Now we need the same thing down here as inside the sine…

$$= \pi\frac{\tau}{2\pi}\frac{2\sin\left(\pi\frac{\omega\tau}{2\pi}\right)}{\pi\frac{\tau}{2\pi}\omega} = \tau\frac{\sin\left(\pi\frac{\omega\tau}{2\pi}\right)}{\pi\frac{\omega\tau}{2\pi}} = \tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2\pi}\right)$$

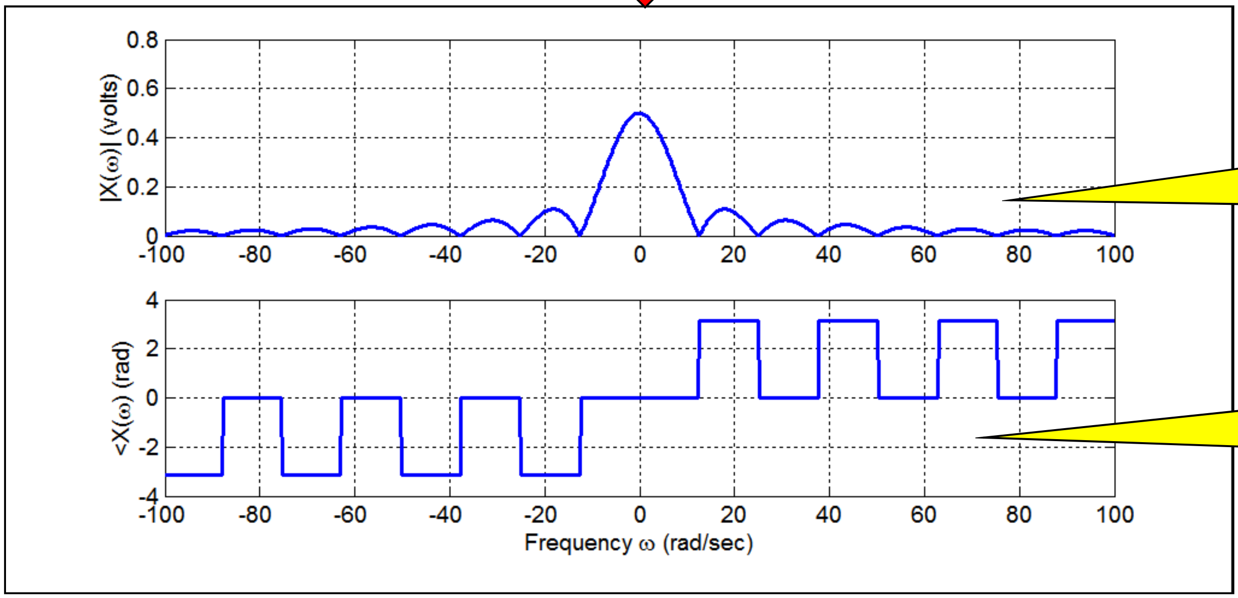
$$\Rightarrow \quad P_\tau(\omega) = \tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2\pi}\right)$$

FT of Rect. Pulse = Sinc Function



Time-Domain View

Frequency-Domain View



Tells what **amplitude** is needed at each frequency

Tells what **phase** is needed at each frequency