EECE 301
Signals & Systems
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Note Set #10

• C-T Signals: Circuits with Periodic Sources
Solving Circuits with Periodic Sources

FS makes it easy to find the response of an RLC circuit to a periodic source!
- Use the FS to convert the source into a sum of sinusoids
- Do phasor analysis for each of the input sinusoids (think superposition!)
- Add up the sinusoidal responses to get the output signal

Example: In electronics you have seen (or will see) how to use diodes and an RC filter circuit to create a DC power supply:

60Hz Sine wave
(Period = 16.67 ms)

\[ x(t) = 60 \sin(240 \pi t) \]

Periodic!! Think Fourier Series!!

Obviously we can’t do this for all infinitely many terms… but we can do it for enough… and if we do it numerically it is not hard!

\[ \omega_0 = 240 \pi \text{ rad/sec} \]

\[ T = 8.33 \text{ ms} \]
**Progression of Ideas**

**Periodic Source**

\[ x(t) \]

\[ R \]

\[ C \]

\[ y(t) \]

\[ T = 8.33 \text{ms} \]

**Periodic Source as FS**

\[ x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \]

\[ R \]

\[ C \]

\[ y(t) \]

**k^{th} Phasor Term of FS**

\[ c_k \text{ @ } k\omega_o \]

\[ Z_C = \frac{1}{jk\omega_o C} \]

\[ d_k = \left[ \frac{Z_C}{Z_C + R} \right] \]

\[ c_k = \left[ \frac{1/jk\omega_o C}{1/jk\omega_o C + R} \right] \]

**Need to find the \( c_k \) values… numerically or analytically**
For this scenario we can find the $c_k$ analytically…

The equation for the FS coefficients is:

$$c_k = \frac{1}{T} \int_0^T x(t) e^{-j k \omega_0 t} \, dt$$

$$\omega_0 = \frac{2\pi}{T}$$

Over this interval:

$$x(t) = A \sin \left( \frac{\pi}{T} t \right) \quad 0 \leq t \leq T$$

$$c_k = \frac{1}{T} \int_0^T A \sin \left( \frac{\pi}{T} t \right) e^{-j k \left( \frac{2\pi}{T} \right) t} \, dt$$

Now apply Calc I ideas to evaluate….

Change of variable: $\tau = \frac{\pi}{T} t$

$$c_k = \frac{A}{\pi} \int_0^\pi \sin(\tau) e^{-j k 2\tau} \, d\tau$$

Use a Table of Integrals and do some algebra & trig to get:

$$c_k = \frac{2A}{\pi(1 - 4k^2)}$$

FS coefficient for full-wave rectified sine wave of amplitude $A$
So the two-sided spectrum after the rectifier:

\[ |C_k| \]

\[ \angle C_k \]

Now we can use Parseval’s Theorem to determine how many terms we need in our approximation for the source…

\[ P = \frac{1}{T} \int_0^T A^2 \sin^2 \left( \frac{\pi}{T} t \right) dt = \frac{A^2}{\pi} \int_0^\pi \sin^2 (\tau) d\tau = \frac{A^2}{\pi^2} \frac{\pi}{2} = \frac{A^2}{2} \]

\[ P_{\text{approx}} = \sum_{k=-K}^{K} |c_k|^2 = \sum_{k=-K}^{K} \left| \frac{2A}{\pi(1-4k^2)} \right|^2 = \frac{4A^2}{\pi^2} \sum_{k=-K}^{K} \left| \frac{1}{1-4k^2} \right|^2 \]
We can look at the ratio of these two as a good measure:

\[
\frac{P_{\text{approx}}}{P} = \frac{4A^2}{\pi^2} \sum_{k=-K}^{K} \left| \frac{1}{(1 - 4k^2)} \right|^2 = \frac{8}{2} \sum_{k=-K}^{K} \left| \frac{1}{(1 - 4k^2)} \right|^2
\]

Numerically evaluating this for different \( K \) values shows that \( K = 10 \) retains more than 99.99% of the power. So we can use that value.

So… our numerical approach is now this:
1. Numerically evaluate \( c_k \) for \( k = -10 \) to 10
2. Numerically convert them into the \( d_k \) phasors
3. Convert the phasors into corresponding FS sinusoidal terms and add them up

We’ll do this for:
- \( A = 10 \) volts
- \( R = 100 \ \Omega \)
- \( C = 1000 \ \mu\text{F} \)
```
wo=240*pi;  % Set fund freq
fo=wo/(2*pi);  % convert to Hz
T = 2*pi/wo;  % compute period
K=10;  % Set number of terms
kv=(-K):K;  % set vector of k indices
A=10;  % set amplitude of input
R=100;  % set resistance
C=1000e-6;  % set capacitance

ck=(2*A/pi)./(1-4*(kv.^2));  % compute the input FS coefficents

dk=(1./(1+j*kv*wo*R*C)).*ck;  % compute the output FS coefficents

Fs = 4*K*fo;  % Compute sampling rate (set here to twice the minimum value of 2Kfo)
Ts = 1/Fs;  % Compute sample spacing
t = (-3*T):Ts:(3*T);

x_apprx = zeros(size(t));  % sets up vector of zeros as first “partial sum”
for k = (-K):K  % loop through “all” coefficients
    x_apprx = x_apprx + ck(k+K+1)*exp(j*k*wo*t);  % Add current term to partial sum
end
x_apprx = real(x_apprx);

y_apprx = zeros(size(t));  % sets up vector of zeros as first “partial sum”
for k = (-K):K  % loop through “all” coefficients
    y_apprx = y_apprx + dk(k+K+1)*exp(j*k*wo*t);  % Add current term to partial sum
end
y_apprx = real(y_apprx);  % theory says imaginary parts cancel… so enforce this in case
    % of numerical round-off issues

figure(1); plot(t,x_apprx,'r',t,y_apprx,'g--'); xlabel('time (seconds)'); ylabel('Input and Output (volts)'); grid
figure(2); subplot(2,1,1); stem(kv,abs(ck)); subplot(2,1,2); stem(kv,abs(dk))
```
Rectified sinewave applied to RC circuit

Output of RC circuit: DC level with small wiggle
\[ d_k = \left[ \frac{1}{1 + jk\omega_0 RC} \right] c_k \]

**Changed via Multiplication!**

**Effect of RC circuit:** Make all non-DC FS coefficients negligible

**Multiplicative factor has small magnitude here!**
Big Idea: “Frequency Response”

Input: \[ x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j k \omega_0 t} \]

Output: \[ y(t) = \sum_{k=-\infty}^{\infty} d_k e^{j k \omega_0 t} \]

Linear Circuit

Input’s FS Coefficients

Output’s FS Coefficients

\[ d_k = H(k \omega_0) c_k \]

\[ H(\omega) \] is the “Frequency Response” of the Circuit

How to find the Frequency Response of a Circuit...

- Assume arbitrary phasor \( X \) with frequency \( \omega \)
- Analyze circuit to find output phasor \( Y \)
  - It will always take this multiplicative form: \( Y = H(\omega) X \)
  - All impedances are evaluated at the arbitrary frequency \( \omega \)
- The frequency response function \( H(\omega) \) is the thing that multiplies \( X \)