Binghamton University
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EECE 301
Signals & Systems
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Note Set #7

- C-T Signals: Three Forms of Fourier Series

Fourier Series Motivation

"Fourier Series" allows us to write "virtually any" real-world *PERIODIC* signal as a sum of sinusoids with appropriate amplitudes and phases.

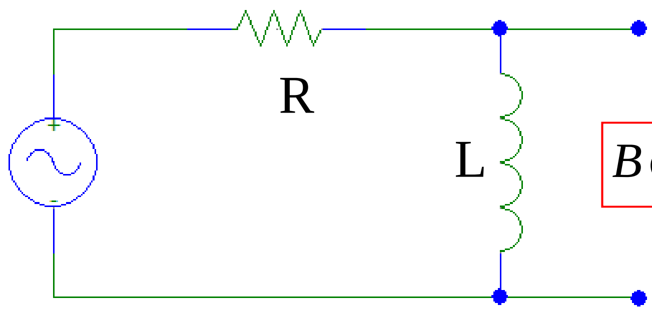
So… we can think of "building a periodic signal from sinusoidal building blocks".

Later we will extend that idea to also build many non-periodic signals from sinusoidal building blocks!

Thus, it is very common for engineers to think about "virtually any" signal as being made up of "sinusoidal components".

Q: Why all this attention to **sinusoids**?
A: Recall from Circuits… "sinusoidal analysis" of RLC circuits:
Fundamental Result: Sinusoid In ⇒ Sinusoid Out
(Same Frequency, Different Amplitude & Phase)

$A\cos(\omega_0 t + \phi), \ -\infty < t < \infty$



$B\cos(\omega_0 t + \theta), \ -\infty < t < \infty$

This "sinusoid in, sinusoid out" result holds for Constant-Coefficient, Linear Differential Equations as well as any LTI system. We'll only motivate this result for this Diff. Eq.:

$$\ddot{y}(t) + a_1\dot{y}(t) + a_0 y(t) = x(t)$$

If the input $x(t)$ is a sinusoid $A\cos\left(\omega_0 t + \phi\right), \ -\infty < t < \infty$

… then the solution $y(t)$ must be such that it and its derivatives can be combined to give the input sinusoid.

So… suppose the solution is $y(t) = B\cos\left(\omega_0 t + \theta\right), \ -\infty < t < \infty$

$$\omega_o^2 B\cos(\omega_o t + \theta) + a_1\omega_o B\sin(\omega_o t + \theta) + a_0 B\cos(\omega_o t + \theta) = A\cos(\omega_o t + \phi)$$

By slogging through lots of algebra and trig identities we can show this can be met with a proper choice of B and θ.

But it makes sense that to add up to a sinusoid we'd need all the terms on the left to be sinusoids of some sort!!!

So… we have reason to believe this:

Fundamental Result: Sinusoid In ⇒ Sinusoid Out

(Same Frequency, Different Amplitude & Phase)

Now… if our input is the linear combination of sinusoids:

$$x(t) = A_1 \cos\left(\omega_1 t + \phi_1\right) + A_2 \cos\left(\omega_2 t + \phi_2\right) + A_3 \cos\left(\omega_3 t + \phi_3\right) + \cdots, \; -\infty < t < \infty$$

By linearity (i.e., superposition) we know that we can simply handle each term separately… and we know that each input sinusoid term gives an output sinusoid term:

$$y(t) = B_1 \cos\left(\omega_1 t + \theta_1\right) + B_2 \cos\left(\omega_2 t + \theta_2\right) + B_3 \cos\left(\omega_3 t + \theta_3\right) + \cdots, \; -\infty < t < \infty$$

So… breaking a signal into sinusoidal parts makes the job of solving a Diff. Eq. EASIER!! (This was Fourier's big idea!!)

But…. What kind of signals can we use this trick on?

Or in other words…

What kinds of signals can we build by adding together sinusoids??!!!

So… Let's Explore What We Can Build with Sinusoids!

Let ω_0 be some given "fundamental" frequency

Q: What can I build from building blocks that looks like:

$$A_k \cos(\underbrace{k\omega_o} + \theta_k) \ ?$$

Only frequencies that are ***integer*** multiples of ω_o

Ex.: ω_o = 30 rad/sec then consider 0, 30 60, 90, …

We can explore this by choosing a few different cases of values for the A_k and θ_k

On the next slide we limit ourselves to looking at three cases where we limit ourselves to having only three terms…

For this example let $\omega_0 = 2\pi$ rad/sec and look at a sum for k = 1, 2, 3:

$$x(t) = A_1 \cos\left(2\pi t + \phi_1\right) + A_2 \cos\left(2 \times 2\pi t + \phi_2\right) + A_3 \cos\left(3 \times 2\pi t + \phi_3\right)$$

$A_1 = 1.0 \quad \theta_1 = 0$

$A_2 = 0.5 \quad \theta_2 = \pi/4$

$A_3 = 0.5 \quad \theta_3 = \pi/2$

Equal

$A_1 = 0.1 \quad \theta_1 = 0$

$A_2 = 1.0 \quad \theta_2 = \pi/4$

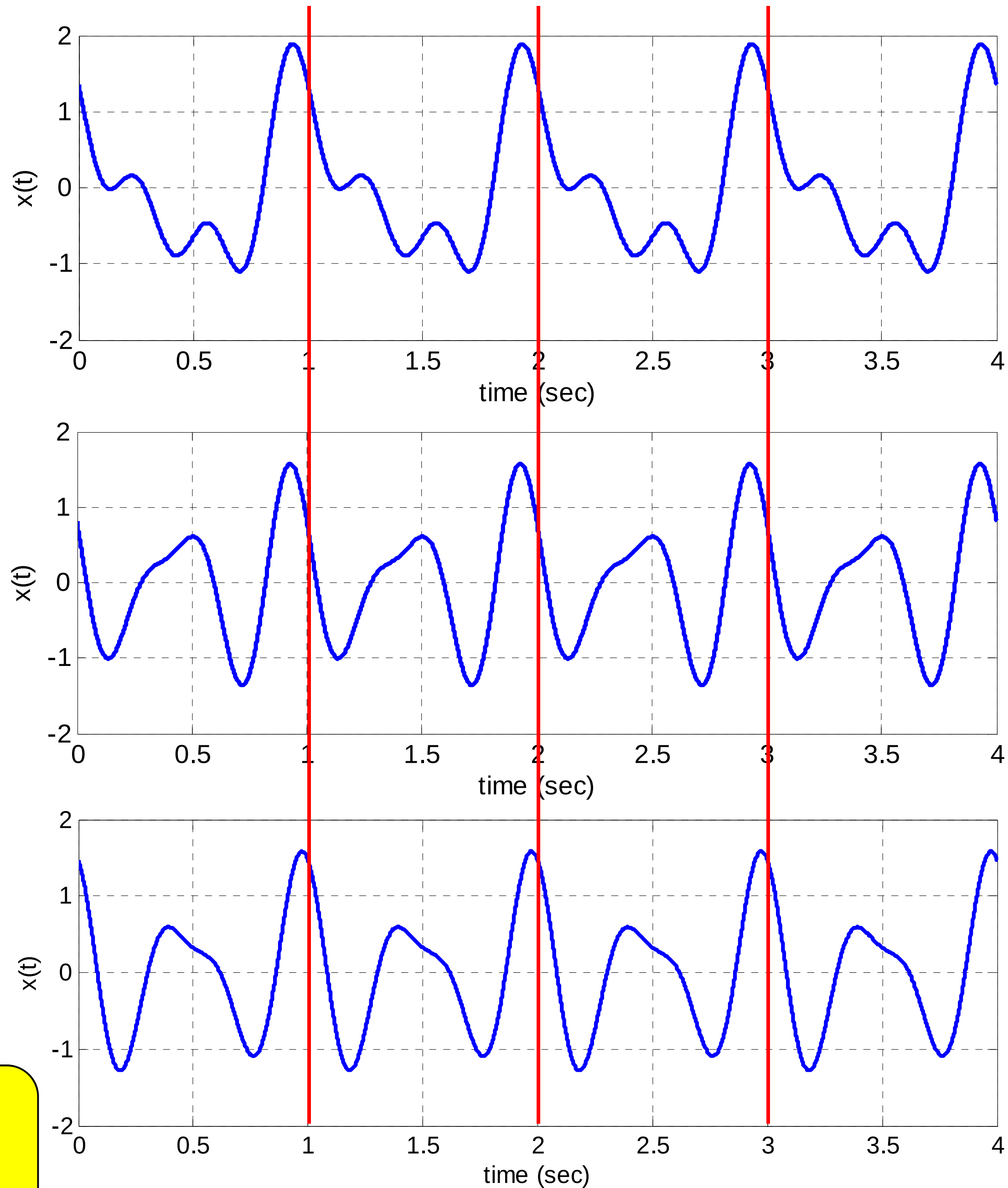
$A_3 = 0.5 \quad \theta_3 = \pi/2$

Equal

$A_1 = 0.1 \quad \theta_1 = 0$

$A_2 = 1.0 \quad \theta_2 = \pi/7$

$A_3 = 0.5 \quad \theta_3 = \pi/14$



Note:

1. **All are periodic with period of 1s**
2. **All are "centered" vertically @ 0**

In one period: Area Above = Area Below

Why do these all have period of 1 s???

$$x(t) = A_1 \cos\left(2\pi t + \phi_1\right) + A_2 \cos\left(2 \times 2\pi t + \phi_2\right) + A_3 \cos\left(3 \times 2\pi t + \phi_3\right)$$

Repeats every 1 s

Repeats every 1/2 s
… so it also repeats every 1 s

Repeats every 1/3 s
… so it also repeats every 1 s

This motivates the following general statement:

> A sum of sinusoids with frequencies that are integer multiples of some lowest "fundamental" frequency ω_o will give a periodic signal with period $T = 2\pi/\omega_o$ seconds.

So… we can now think about adding together any number of harmonically-related sinusoids… even infinitely many!

$$x(t) = \sum_{k=1}^{\infty} A_k \cos\left(k\omega_o t + \phi_k\right), \quad -\infty < t < \infty$$

i.e., all frequencies are an *integer* multiple of fund. freq. ω_o

Why are these all centered vertically @ 0???

$$x(t) = \underbrace{A_1 \cos\left(2\pi t + \phi_1\right)}_{\text{Centered @ 0}} + \underbrace{A_2 \cos\left(2 \times 2\pi t + \phi_2\right)}_{\text{Centered @ 0}} + \underbrace{A_3 \cos\left(3 \times 2\pi t + \phi_3\right)}_{\text{Centered @ 0}}$$

This motivates the following general statement:

> Unless we have a constant term added, a sum of sinusoids (with frequencies at ω_o, $2\omega_o$, $3\omega_o$, …) will be centered vertically at 0

So… we can now add a constant term

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos\left(k\omega_o t + \phi_k\right), \; -\infty < t < \infty$$

Note: for $k = 0$ we have $A_0 \cos(0 \times \omega_o t) = A_0$ so we can think of the constant term as a cosine with frequency = 0 and phase = 0

Fourier Series… A Way to Build a Periodic Signal

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos\left(k\omega_o t + \phi_k\right), \; -\infty < t < \infty$$

This signal has Period $T = 2\pi/\omega_o$

Big Idea: We can think of (virtually) any real-world **periodic** signal as being made up of (possibly infinitely) many sinusoids whose frequencies are all an integer multiple of a fundamental frequency ω_o.

(We won't prove that here… but it can be proven and the proof is in the book)

Once we set ω_o all we have to do is specify all the amplitudes (A_k) and phases (θ_k) and we get some periodic signal with period $T = 2\pi/\omega_o$.

But… if we are **GIVEN a periodic signal** how do we determine the correct:

- Fundamental Frequency ω_o (rad/sec)
- Amplitudes (A_k)
- Phases (θ_k)

Easy: $\omega_o = 2\pi/T$

Need to Learn How!!

Three Forms of Fourier Series

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos\left(k\omega_o t + \phi_k\right)$$

"Amplitude & Phase" Form

The equation above is just one of three (totally equivalent!) different forms of the Fourier Series.

Each one contains the same information but presents it differently.

Which form you use in a particular setting depends….

- Partly on your preference
- Partly on what you are trying to do

Both of these come with experience…

We can easily find the other two by applying trig identities to the terms in the above form.

Convert to Complex Exponential Form

$$x(t) = A_0 + A_1\cos\left(1\omega_o t + \phi_1\right) + A_2\cos\left(2\omega_o t + \phi_2\right) + \cdots$$

"Amplitude & Phase" Form

Euler's Formula

$$\cos(\theta) = \tfrac{1}{2}\left[e^{j\theta} + e^{-j\theta}\right]$$

$$x(t) = \underbrace{A_0}_{\triangleq c_0} + \left[\underbrace{\frac{A_1}{2}e^{j\phi_1}}_{\triangleq c_1} e^{j1\omega_o t} + \underbrace{\frac{A_1}{2}e^{-j\phi_1}}_{\triangleq c_{-1}} e^{j(-1)\omega_o t}\right] + \left[\underbrace{\frac{A_2}{2}e^{j\phi_2}}_{\triangleq c_2} e^{j2\omega_o t} + \underbrace{\frac{A_2}{2}e^{-j\phi_2}}_{\triangleq c_{-2}} e^{j(-2)\omega_o t}\right] + \cdots$$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

"Complex Exponential" Form

Convert to Sine-Cosine Form

$$x(t) = A_0 + A_1\cos\left(1\omega_o t + \phi_1\right) + A_2\cos\left(2\omega_o t + \phi_2\right) + \cdots$$

"Amplitude & Phase" Form

Trig Identity

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(B) - \sin(\alpha)\sin(\beta)$$

$$x(t) = \underbrace{A_0}_{\triangleq a_0} + \left[\underbrace{A_1\cos\left(\phi_1\right)}_{\triangleq a_1}\cos\left(1\omega_o t\right) \underbrace{- A_1\sin\left(\phi_1\right)}_{\triangleq b_1}\sin\left(1\omega_o t\right)\right]$$

$$+ \left[\underbrace{A_2\cos\left(\phi_2\right)}_{\triangleq a_2}\cos\left(2\omega_o t\right) \underbrace{- A_2\sin\left(\phi_2\right)}_{\triangleq b_2}\sin\left(2\omega_o t\right)\right] + \cdots$$

$$x(t) = a_0 + \sum_{k=1}^{\infty}\left[a_k\cos(k\omega_0 t) + b_k\sin(k\omega_0 t)\right]$$

"Sine-Cosine" Form

Three (Equivalent) Forms of FS and Their Relationships

Best for "thinking about real-world ideas"

Trig Form: Amplitude & Phase

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$$

$A_0 = c_0$

$A_k = 2|c_k|$

$\theta_k = \angle c_k$

$\Big\} \; k = 1, 2, 3, \ldots$

$c_0 = A_0$

$c_k = \frac{1}{2} A_k e^{j\theta_k}$

$c_{-k} = \frac{1}{2} A_k e^{-j\theta_k}$

$\Big\} \; k = 1, 2, 3, \ldots$

Best for "doing math"
(c_k are like phasors!!)

Exponential Form

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

$A_0 = a_0$

$A_k = \sqrt{a_k^2 + b_k^2}$

$\theta_k = \tan^{-1}\left(\frac{-b_k}{a_k}\right)$

$a_0 = c_0$

$a_k = A_k \cos(\theta_k)$

$b_k = -A_k \sin(\theta_k)$

$c_0 = a_0$

$c_k = \frac{1}{2}(a_k - jb_k)$

$c_{-k} = \frac{1}{2}(a_k + jb_k)$

$\Big\} \; k = 1, 2, 3, \ldots$

$a_0 = c_0$

$a_k = 2\,\mathrm{Re}\{c_k\}, \quad k = 1, 2, 3, \ldots$

$b_k = -2\,\mathrm{Im}\{c_k\}, \quad k = 1, 2, 3, \ldots$

Best for some "special scenarios"

Trig Form: Sine-Cosine

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left[a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)\right]$$

Example: Consider $x(t) = \cos(t) + 0.5\cos(4t + \pi/3) + 0.25\cos(8t + \pi/2)$

which is already in **Amp-Phase Form** of the Fourier Series with $\omega_0 = 1$:

$A_1 = 1$ $\quad A_4 = 0.5$ $\quad A_8 = 0.25$ $\quad$ (all other A_k are 0)

$\theta_1 = 0$ $\quad \theta_4 = \pi/3$ $\quad \theta_8 = \pi/2$

Using the conversion results on the previous slide we can re-write this in **Complex Exponential Form** of the FS as:

$c_1 = 0.5$ $\quad c_4 = 0.25e^{j\pi/3}$ $\quad c_8 = 0.125e^{j\pi/2}$ $\quad$ (all other c_k are 0)

$c_{-1} = 0.5$ $\quad c_{-4} = 0.25e^{-j\pi/3}$ $\quad c_{-8} = 0.125e^{-j\pi/2}$

$$x(t) = \left[0.5e^{jt} + 0.5e^{-jt}\right] + \left[0.25e^{j\pi/3}e^{j4t} + 0.25e^{-j\pi/3}e^{-j4t}\right] + \left[0.5e^{j\pi/2}e^{j8t} + 0.5e^{-j\pi/2}e^{-j8t}\right]$$

Using the conversion results on the previous slide we can re-write this in **Sine-Cosine Form** of the FS as:

$a_1 = 1$ $\quad a_4 = 0.25$ $\quad a_8 = 0$ $\quad$ (all other a_k, b_k are 0)

$b_1 = 0$ $\quad b_4 = 0.43$ $\quad b_8 = 0.25$

$$x(t) = \left[\cos(t)\right] + \left[0.25\cos(4t) + 0.43\sin(4t)\right] + \left[0.25\sin(8t)\right]$$