Binghamton University
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EECE 301
Signals & Systems
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Note Set #6

- Sinusoidal Time Functions
- Complex-Valued Sinusoidal Time Functions
- Sampling Sinusoids: DT Sinusoids

Sinusoidal Time Function

A sinusoid is completely defined by its three parameters:

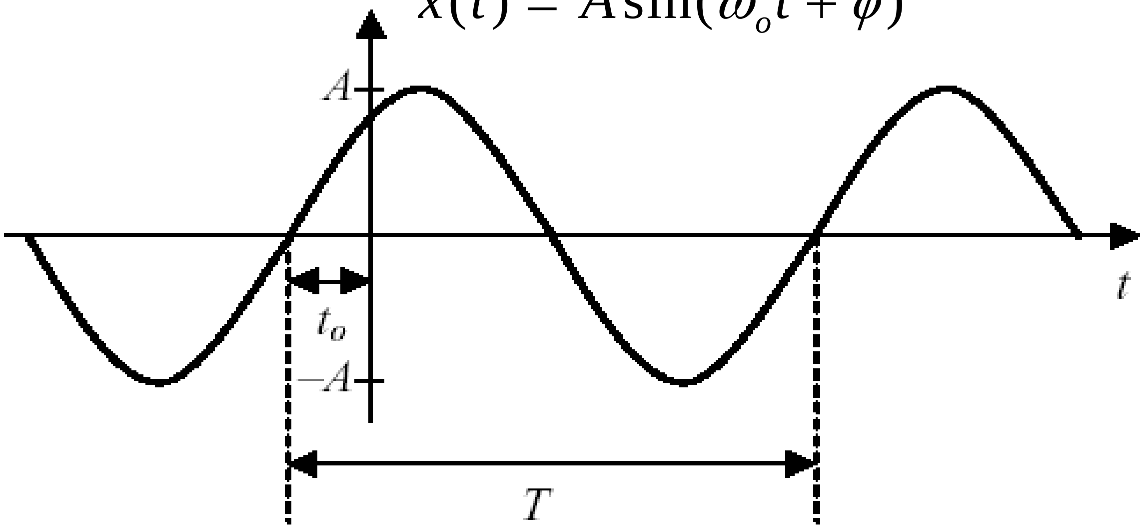
- **Amplitude** A (for us typically in volts or amps but could be other unit)
- **Frequency** ω_o in rad/sec (not Hz!)
- **Phase** ϕ in rad (not degrees!)

$$x(t) = A\sin(\omega_o t + \phi)$$

(rad/sec)×sec + rad = rad

(Similar for cosine)

$$x(t) = A\sin(\omega_o t + \phi)$$



T is the Period of the sinusoid… it is related to the frequency ω_o

t_o is a time shift… it is related the phase ϕ

Relation between Period and Frequency

$$x(t) = A\sin(\omega_o t + \phi)$$

If T is the period… the sine's argument must change by 2π as t goes from 0 to T

$$(\omega_o T + \phi) - (\omega_o 0 + \phi) = \omega_o T \quad \Rightarrow \quad \omega_o T = 2\pi$$

$$\Rightarrow \quad \boxed{T = \frac{2\pi}{\omega_o}} \quad \text{or} \quad \boxed{\omega_o = \frac{2\pi}{T}}$$

(Can always check via units!)

Relation between Phase and Time Shift

Phase shift (often just shortened to phase) shows up <u>explicitly</u> in the equation but shows up <u>implicitly</u> in the plot as a time shift (because the plot is vs. time).

We can write the time shift of a function by replacing t by $t + t_o$. So start with an unshifted sine function $A\sin(\omega_o t)$ and time shift it by t_o. Then we get:

$$A\sin(\omega_o(t + t_o)) = A\sin(\omega_o t + \underbrace{\omega_o t_o})$$

$$\Rightarrow \quad \boxed{\phi = \omega_o t_o}$$

(Unit-wise this makes sense!!!)

This is important in the lab…

You can measure time shift on the scope but need phase shift for the math

Relation between Radian Frequency and Cyclic Frequency

No different from any unit conversion: yards to meters, dollars to euros, etc!

Frequency is nothing more than the rate of change of angle… and the unit used depends on the unit used for angles.

Radian Frequency ω_o: angle is measured in radians… rad/sec
Cyclic Frequency f_o: angle is measured in cycles… cycles/sec = Hz

There are 2π radians/cycle so… (f_o cycles/sec)$\times$ (2π radians/cycle) gives rad/sec

$$\Rightarrow \quad \omega = 2\pi f$$

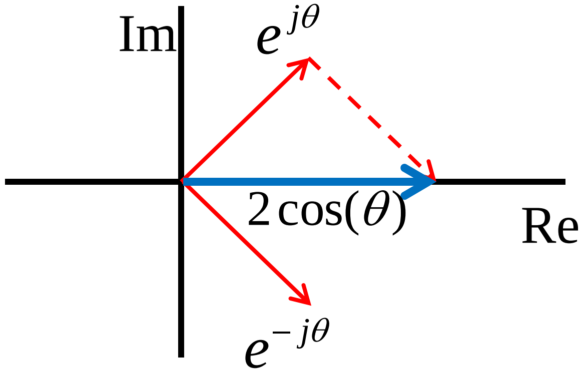
Complex Sinusoidal Time Function

In many cases it is desirable to write a real-valued sinusoid in terms of "complex-valued sinusoids". This is a math trick that – believe it or not! – makes things ***easier*** to work with!!!

$$x(t) = A\cos(\omega_o t + \phi) = \frac{A}{2}\left[e^{j(\omega_o t+\phi)} + e^{-j(\omega_o t+\phi)}\right]$$

This comes from Euler's Formula:

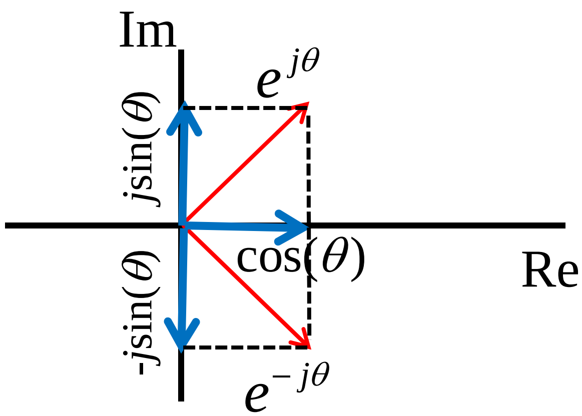
$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$$



Another form of Euler's Formula:

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$e^{-j\theta} = \cos(\theta) - j\sin(\theta)$$

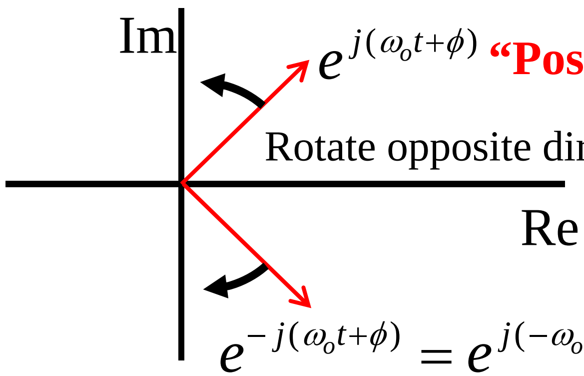


<u>Exploring the Complex Sinusoidal Terms</u>

$$x(t) = A\cos(\omega_o t + \phi) = \frac{A}{2}\left[\underbrace{e^{j(\omega_o t + \phi)} + e^{-j(\omega_o t + \phi)}} \right]$$

Two complex values with opposite angles

Imaginary part always cancels!



$e^{j(\omega_o t + \phi)}$ **"Positive Frequency" Term**

Rotate opposite directions… due to negative sign

$e^{-j(\omega_o t + \phi)} = e^{j(-\omega_o t - \phi)}$ **"Negative Frequency" Term**

Here is a link to a Quicktime movie of these rotating…

http://www.cic.unb.br/~mylene/PSMM/DSPFIRST/chapters/2sines/demos/phasors/graphics/phasorsn.mov

Link to another Web Demo of this…

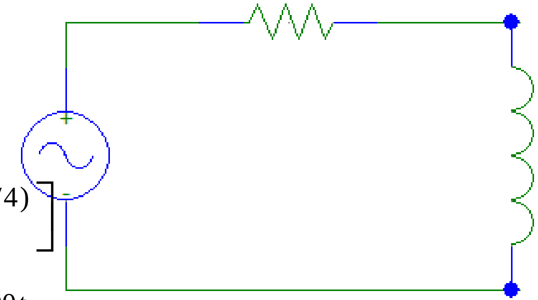
1. Open the web page
2. Click on the box at the top labeled Two

<u>Example of Usefulness of the Complex Sinusoidal Terms</u>

(or… "Phasors Revisited"!)

Suppose you have the RL circuit shown below with a sinusoidal input voltage and you want to find the current $i(t)$ (through R or through L… they *are* the same!)

R = 10Ω



L = 2H

Values picked for ease not to be realistic!

$$x(t) = 6\cos(100t + \pi/4)$$
$$= \frac{6}{2}\left[e^{j(100t+\pi/4)} + e^{-j(100t+\pi/4)}\right]$$
$$= 3e^{j\pi/4}e^{j100t} + 3e^{-j\pi/4}e^{-j100t}$$

By KVL: $x(t) = v_R(t) + v_L(t)$ Then… use the "device rules":

$$v_R(t) = Ri(t)$$
$$v_L(t) = L\frac{di(t)}{dt}$$

$$L\frac{di(t)}{dt} + Ri(t) = x(t)$$

A differential equation!

Now since we've written the input as a sum of two things we can use linearity (aka superposition) and consider each term alone… then sum results.

So… subbing in only the positive frequency term: $L\frac{di_p(t)}{dt} + Ri_p(t) = 3e^{j\pi/4}e^{j100t}$

Thus… we need an $i(t)$ that combines with its derivative to give an exponential…
We can guess that $i(t)$ must be a similar exponential:

$$i_p(t) = A_p e^{j\theta_p} e^{j100t}$$

$$\frac{di_p(t)}{dt} = j100 A_p e^{j\theta_p} e^{j100t}$$

So plugging in our guess gives

$$2\left(j100 A_p e^{j\theta_p} e^{j100t}\right) + 10\left(A_p e^{j\theta_p} e^{j100t}\right) = 3e^{j\pi/4}e^{j100t}$$

Combining terms:

$$\left(j200 A_p e^{j\theta_p} + 10 A_p e^{j\theta_p}\right) e^{j100t} = 3e^{j\pi/4}e^{j100t}$$

Must be Equal!

$$A_p e^{j\theta_p} = \frac{3e^{j\pi/4}}{(10 + j200)} = \frac{3e^{j\pi/4}}{200.25e^{j1.52}} = 0.015e^{-j0.73}$$

Convert to Polar!

So… similarly for the <u>negative</u> frequency term: $L\frac{di_n(t)}{dt} + Ri_n(t) = 3e^{-j\pi/4}e^{-j100t}$

Now…We can guess that $i_n(t) = A_n e^{j\theta_n} e^{-j100t}$ $\quad \frac{di_n(t)}{dt} = -j100 A_n e^{j\theta_n} e^{-j100t}$

So plugging in our guess and solving like before gives

$$A_n e^{j\theta_n} = 0.015e^{+j0.73} \quad \text{Compare} \quad A_p e^{j\theta_p} = 0.015e^{-j0.73}$$

Same magnitude, opposite sign phase… that will always happpen!!
So…. we don't really need to do the negative part… we can "guess" it!

So finally we have: $i(t) = i_p(t) + i_n(t)$

$$= 0.015e^{-j0.73}e^{j100t} + 0.015e^{j0.73}e^{-j100t}$$

$$= \frac{0.030}{2}\left[e^{j(100t-0.73)} + e^{-j(100t-0.73)}\right]$$

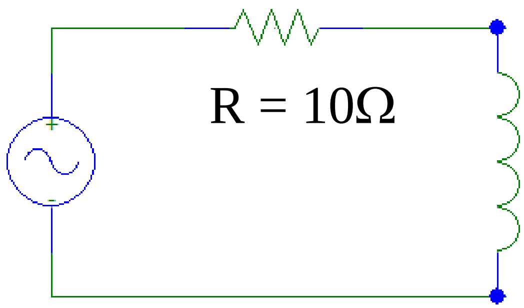
$$= 0.030\cos(100t - 0.73)$$

That looked like quite a bit of work… where is the advantage of the complex view??

1. Try re-doing this by directly using the cosine
   - It can be done but it requires lots of trig identities
2. More importantly… this leads to the idea of phasors
   - We saw that we only need to the positive frequency part
   - We saw that at some point the exp(j100t) term falls out…
   - So… "phasors" capture both those short-cuts
   - And we no longer have to directly deal with the Diff. Eq.

$$x(t) = 6\cos(100t + \pi/4)$$

$$\vec{X} = 6e^{j\pi/4}$$



L = 2H

$$Z_L = j\omega_o L = j100 \times 2$$

Find total impedance: $Z_T = R + Z_L = 10 + j200$

Find current phasor: $\vec{I} = \vec{X} / Z_T = \dfrac{6e^{j\pi/4}}{10 + j200} = \dfrac{6e^{j\pi/4}}{200.25e^{j1.52}} = 0.030e^{-j0.73}$

Ohm's Law

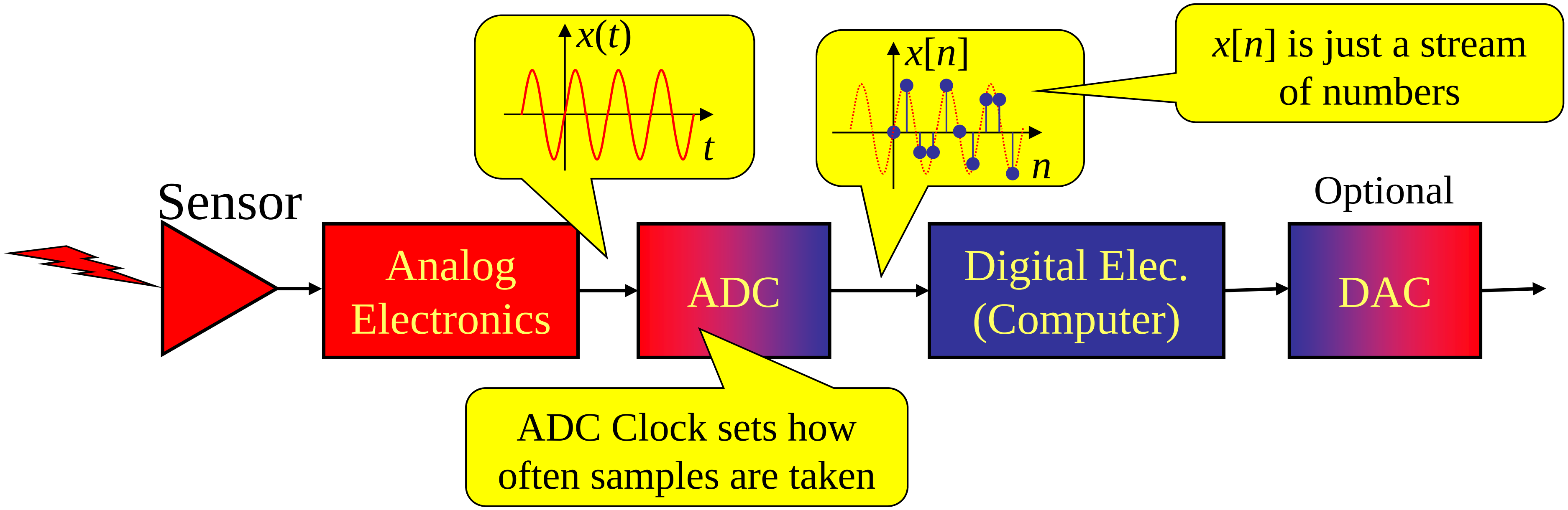
Convert to Sinusoid: $i(t) = 0.030\cos(100t - 0.73)$

Same!!!

Big Ideas for Complex Sinusoids

1. Euler's formula connects real-world sinusoids to the math-only idea of complex sinusoids
2. Although it seems things are made more complex (pun intended!) this actually simplifies the math…
   - Working with exponentials is always easier than sinusoids!
3. We saw that it leads to being able to dispense with the differential equation
   - We'd like to do this for other signals than only sinusoids
   - Fourier's methods allow us to do just that!!!

Sampling Sinusoids… DT Sinusoids



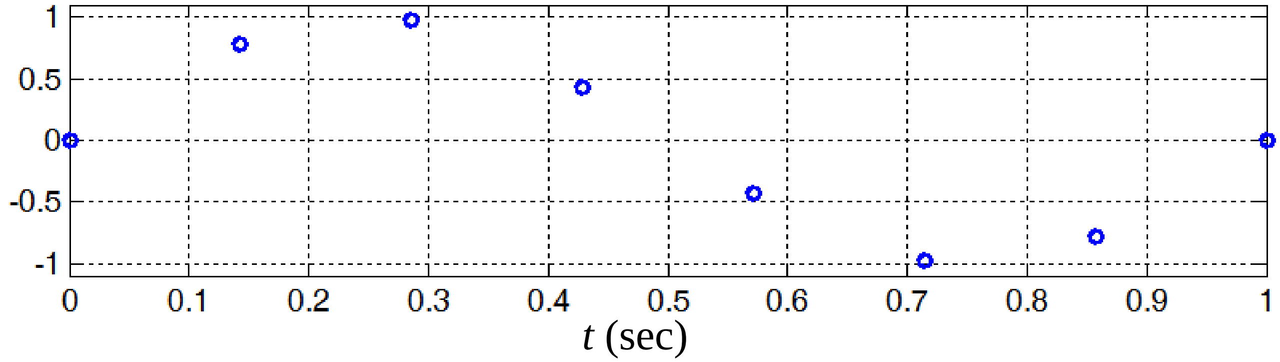
How closely should the samples be spaced??

At first thought we might think we need to have the samples still "look like" the original sinusoid… But that turns out to be excessive, as our theory will show eventually show.

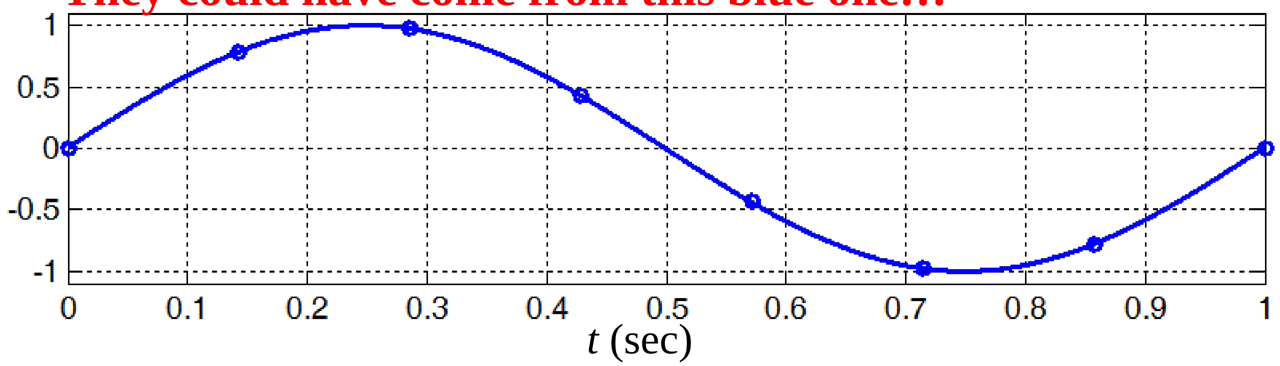
Looking at the samples $x[n]$ above they don't quite really look like a sinusoid… yet they are taken at a rate suitable for most applications!

So… how do we determine how fast we need to sample???

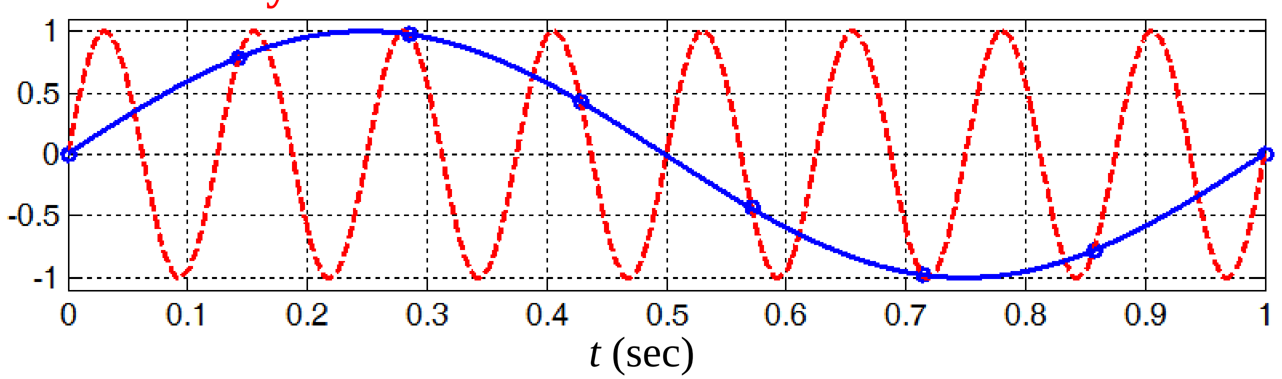
DT Samples…. What CT Sinusoid did they come from????



They could have come from this blue one…



But…They could have come from this RED one!!!



Thus… if we want to be able to tell these two apart we need to sample faster!!

Let T_s be the time spacing between samples… Then $F_s = 1/T_s$ as the "sampling frequency" in samples/sec.

Then if we have a CT sinusoid $x(t) = \cos(2\pi f_o t)$ that is sampled we have

$$x(t) = \cos(2\pi f_o t) \quad \rightarrow \quad x[n] = x(nT_s) = \cos(\underbrace{2\pi f_o T_s}_{\triangleq \Omega_o} n)$$

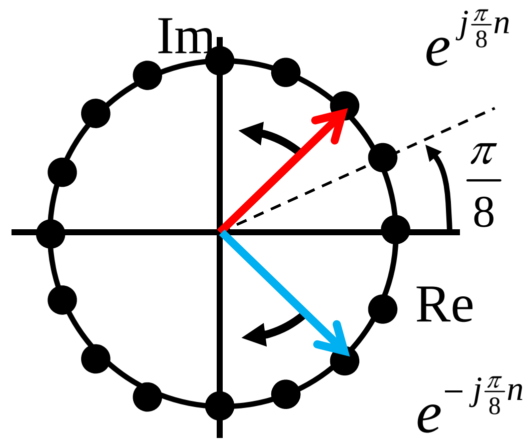
Discrete-Time Sinusoid

$$\Rightarrow \quad x[n] = \cos(\Omega_o n) \qquad \Omega_o \triangleq 2\pi \frac{f_o}{F_s}$$

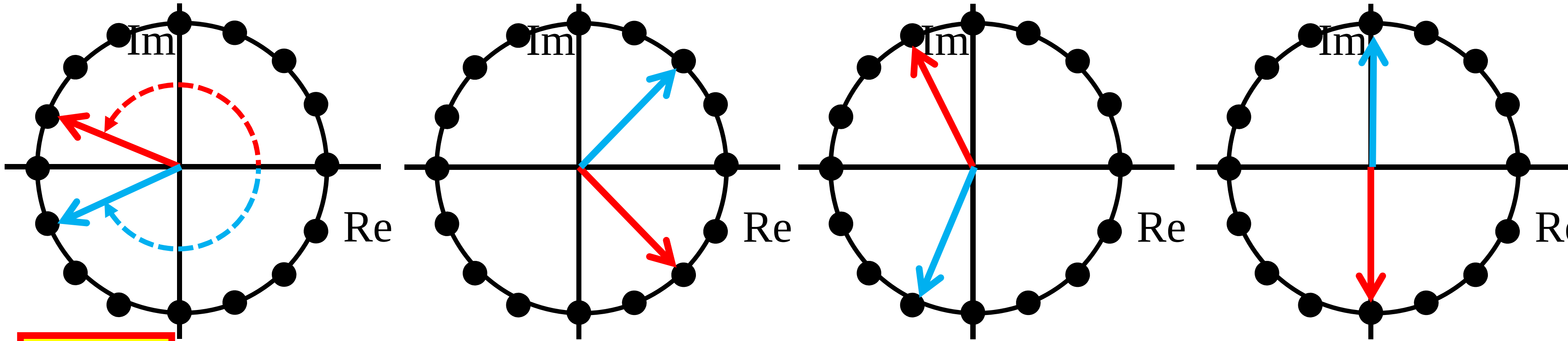
Units are "rad/sample"

So… to help visualize this:

$$\cos(\Omega_o n) = \tfrac{1}{2}\left[e^{j\Omega_o n} + e^{-j\Omega_o n}\right]$$



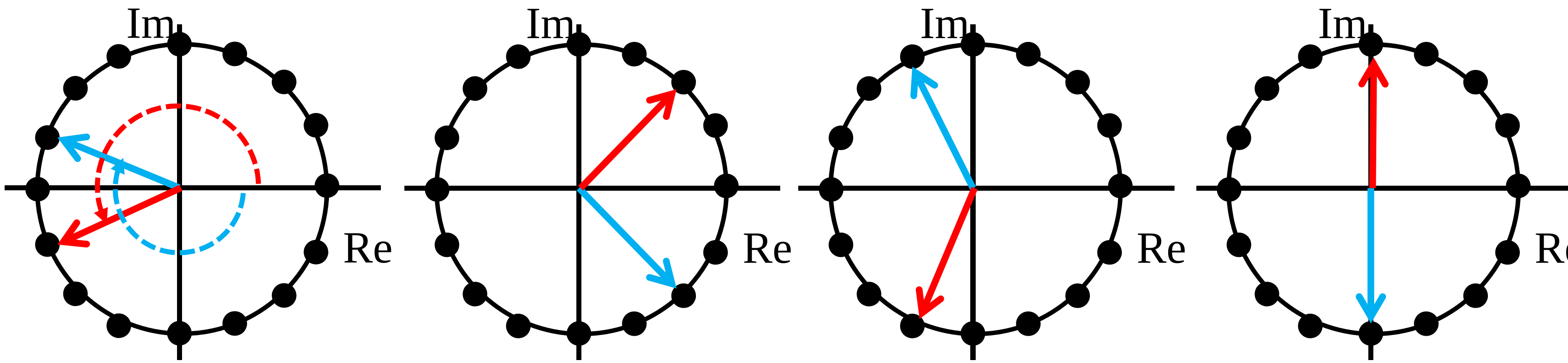
***n* = 1** **_n_ = 2** **_n_ = 3** **_n_ = 4**



$\Omega_o = \frac{7\pi}{8}$

$\Omega_o = \frac{9\pi}{8}$

So… a DT frequency > π rad/sample looks exactly like some other frequency < π. <u>This is called “Aliasing”</u>.



***n* = 1** **_n_ = 2** **_n_ = 3** **_n_ = 4**

So to avoid this "aliasing" when sampling a CT sinusoid to make a DT sinusoid we must require that:

$$f_o < \frac{F_s}{2} \quad \Rightarrow \quad \Omega_o < 2\pi \frac{F_s/2}{F_s} = \pi$$

Thus… for "proper sampling" we need to choose our <u>sampling rate</u> to be more than <u>double</u> the <u>highest frequency we expect</u>!!!

This is consistent with some real-world facts you may know about:

- High-Fidelity Audio contains frequencies up to only about 20 kHz
- CD digital audio has a sampling frequency of F_s = 44.k Hz > 2x20kHz