BINGHAMTON UNIVERSITY

State University of New York

EECE 301
Signals & Systems
Prof. Mark Fowler

Discussion #10

- Laplace Transform Examples

Examples of Solving Differential Equations using LT

Notice how easy this is!

-LT converts the differential equation into an algebraic equation

-We can easily solve an algebraic equation for an output $Y(s)$

-We can do partial fraction expansion

-We can use the LT table

} But, this is where the hard part lies… although it is easy for certain inputs.

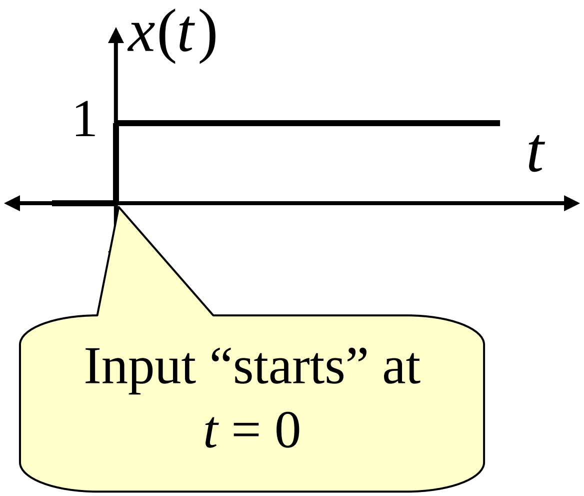
Example 6.29: 2nd Order Differential Equation

Given $$\frac{d^2y(t)}{dt^2}+6\frac{dy(t)}{dt}+8y(t)=2x(t)$$

Assume that the system has ICs given by:

$$y(0^-)=1 \quad and \quad \dot{y}(0^-)=2$$

Find the output $y(t)$ for $t \geq 0$ when the input is $x(t) = u(t)$



Applying the LT to this D.E. $\frac{d^2 y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = 2x(t)$

Gives: $\left[s^2 Y(s) - y(0^-)s - \dot{y}(0^-)\right] + 6\left[sY(s) - y(0^-)\right] + 8Y(s) = 2X(s)$

Solving for $Y(s)$ algebraically gives:

$$Y(s) = \frac{y(0^-)s + \left[\dot{y}(0^-) + 6y(0^-)\right]}{s^2 + 6s + 8} + \frac{2}{s^2 + 6s + 8} X(s)$$

Using the specific IC's gives:

$$Y(s) = \underbrace{\left[\frac{s+8}{s^2 + 6s + 8}\right]}_{\text{IC part}} + \underbrace{\left[\frac{2}{s^2 + 6s + 8}\right]}_{H(s)} X(s)$$

IC part

$H(s)$ "Transfer Function" (notice how this comes directly from the D.E.)

Notice how the IC part can be easily found using PFE and an LT table (For linear, constant coefficient differential equations it will always be like that!)

$$Y(s) = \underbrace{\left[\frac{s+8}{s^2+6s+8}\right]}_{\text{IC part}} + \underbrace{\left[\frac{2}{s^2+6s+8}\right]}_{H(s)} X(s)$$

IC part

$H(s)$ "Transfer Function" (notice how this comes directly from the D.E.)

Notice that the input part may not be easy to do PFE/ILT if the input $X(s)$ is complicated.

But in control systems and sometimes in electronics we are often interested in how the system responds to a step function $\rightarrow$ Called the "step response"

$$\text{Note: } x(t) = u(t) \leftrightarrow X(s) = \frac{1}{s}$$

$$Y(s) = \left[\frac{s+8}{s^2+6s+8}\right] + \left[\frac{2}{s(s^2+6s+8)}\right]$$

IC part Input part

First… compare to this:

$$Ae^{-\zeta\omega_n t}\sin\left[\left(\omega_n\sqrt{1-\zeta^2}\right)t+\phi\right]u(t)$$

$$\text{where: } A = \beta\sqrt{\frac{\left(\frac{\alpha}{\omega_n}-\zeta\omega_n\right)^2}{1-\zeta^2}+1}$$

$$\phi = \tan^{-1}\left(\frac{\omega_n\sqrt{1-\zeta^2}}{\alpha-\zeta\omega_n}\right)$$

$\longleftrightarrow$

$$\beta\frac{s+\alpha}{s^2+2\zeta\omega_n s+\omega_n^2}$$

Is… 0< |ζ| < 1?

And identify:

$$\alpha = 8 \quad \beta = 1$$

$$\omega_n^2 = 8 \quad \Rightarrow \quad \omega_n = 2\sqrt{2}$$

$$2\zeta\omega_n = 6 \quad \Rightarrow \quad \zeta = 6/2\omega_n = 6/4\sqrt{2} = 1.06$$

No!!!
So… factor!!

Factored!!!

$$Y(s) = \left[\frac{s+8}{s^2+6s+8}\right] + \left[\frac{2}{s(s^2+6s+8)}\right] = \left[\frac{s+8}{(s+4)(s+2)}\right] + \left[\frac{2}{s(s+4)(s+2)}\right]$$

IC part — Input part — IC part — Input part

Doing PFE:

```
>> [R,P,K]=residue([1 8],[1 6 8])
R =
   -2
    3
P =
   -4
   -2
K =
    []
```

```
>> [R,P,K]=residue(2,[1 6 8 0])
R =
    0.2500
   -0.5000
    0.2500
P =
    -4
    -2
     0
K =
     []
```

$$\Rightarrow\quad Y(s) = \left(\left[\frac{-2}{s+4}\right] + \left[\frac{3}{s+2}\right]\right) + \left(\left[\frac{0.25}{s+4}\right] + \left[\frac{-0.5}{s+2}\right] + \left[\frac{0.25}{s}\right]\right)$$

IC part — Transient part — SS part

$$Y(s) = \underbrace{\left(\left[\frac{-2}{s+4}\right] + \left[\frac{3}{s+2}\right]\right)}_{\text{IC part}} + \underbrace{\left(\left[\frac{0.25}{s+4}\right] + \left[\frac{-0.5}{s+2}\right]\right)}_{\text{Transient part}} + \underbrace{\left[\frac{0.25}{s}\right]}_{\text{SS part}}$$

Using the LT table:

$$\boxed{e^{-bt}u(t), \quad b \text{ real or complex}} \longleftrightarrow \boxed{\frac{1}{s+b}, \quad b \text{ real or complex}}$$

$1/4e^{0t} = 1/4$

$$y(t) = \underbrace{\left[-2e^{-4t} + 3e^{-2t}\right]}_{\text{IC part}} + \left[\underbrace{1/4e^{-4t} - 1/2e^{-2t}}_{\text{Transient part}} + \underbrace{1/4}_{\text{SS part}}\right], \quad t \geq 0$$

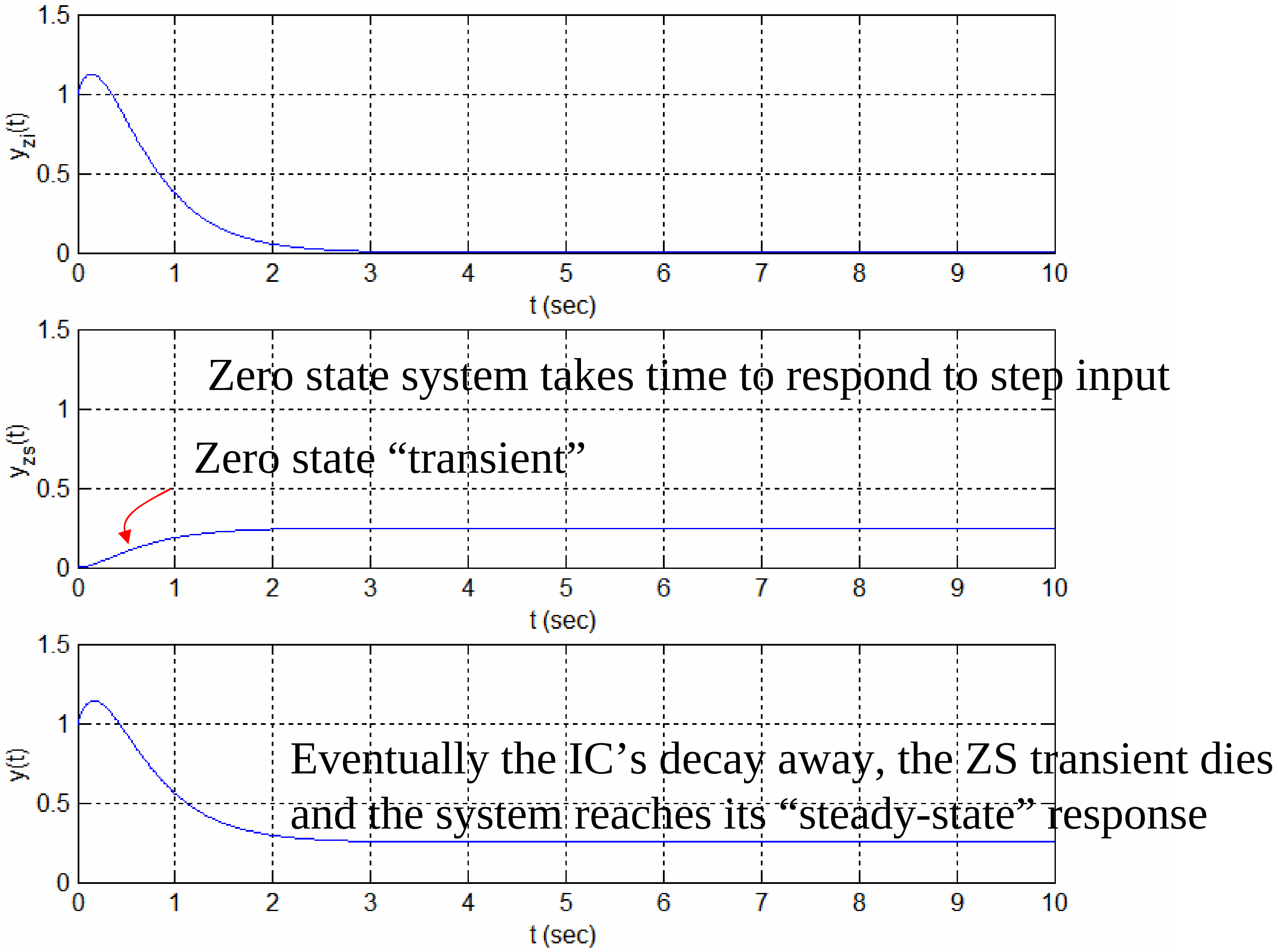
Specifying this means we can leave off the $u(t)$'s

$$y(t) = \underbrace{\left[-2e^{-4t} + 3e^{-2t}\right]}_{\text{IC part}} + \left[\underbrace{1/4e^{-4t} - 1/2e^{-2t}}_{\text{Transient part}} + \underbrace{1/4}_{\text{SS part}}\right], \quad t \geq 0$$

Note: IC part & Transient part have the same kind of decaying parts: e^{-4t} & e^{-2t}
These come from the system's characteristic polynomial!

Note: The book combined $Y(s)$ into one big thing and found $y(t)$ from that.
Same answer…but we cannot see the impact of three parts!

Plots for ex. 6.29



Now Modify Example 6.29: 2nd Order Differential Equation

Was…

$$\frac{d^2y(t)}{dt^2}+6\frac{dy(t)}{dt}+8y(t)=2x(t)$$

$$Y(s)=\underbrace{\left[\frac{s+8}{s^2+6s+8}\right]}_{\text{IC part}}+\underbrace{\left[\frac{2}{s(s^2+6s+8)}\right]}_{\text{Input part}}$$

Now…

$$\frac{d^2y(t)}{dt^2}+\frac{dy(t)}{dt}+8y(t)=2x(t)$$

$$Y(s)=\underbrace{\left[\frac{s+8}{s^2+s+8}\right]}_{\text{IC part}}+\underbrace{\left[\frac{2}{s(s^2+s+8)}\right]}_{\text{Input part}}$$

And identify:

$$\alpha=8 \quad \beta=1$$

$$\omega_n^2=8 \quad \Rightarrow \quad \omega_n=2\sqrt{2}$$

$$2\zeta\omega_n=1 \quad \Rightarrow \quad \zeta=1/2\omega_n=1/4\sqrt{2}=0.18$$

Is… 0< |ζ| < 1? Yes… → Complex Roots… so use one of the 2nd order LT Pairs

$$Y(s) = \left[\frac{s+8}{s^2+s+8}\right] + \left[\frac{2}{s(s^2+s+8)}\right]$$

Do PFE

$$= \left[\frac{s+8}{s^2+s+8}\right] + \left[\frac{0.25}{s} + \frac{-0.125 + j0.0225}{(s+0.5-j2.78)} + \frac{-0.125 - j0.0225}{(s+0.5+j2.78)}\right]$$

$$= \left[\frac{s+8}{s^2+s+8}\right] + \left[\frac{0.25}{s} - 0.25\frac{s+1}{s^2+s+8}\right]$$

Recombine to get back quadratic

$$Ae^{-\zeta\omega_n t}\sin\left[\left(\omega_n\sqrt{1-\zeta^2}\right)t + \phi\right]u(t)$$

$$\text{where}: \quad A = \beta\sqrt{\frac{\left(\frac{\alpha}{\omega_n} - \zeta\omega_n\right)^2}{1-\zeta^2} + 1}$$

$$\phi = \tan^{-1}\left(\frac{\omega_n\sqrt{1-\zeta^2}}{\alpha - \zeta\omega_n}\right)$$

$\longleftrightarrow$

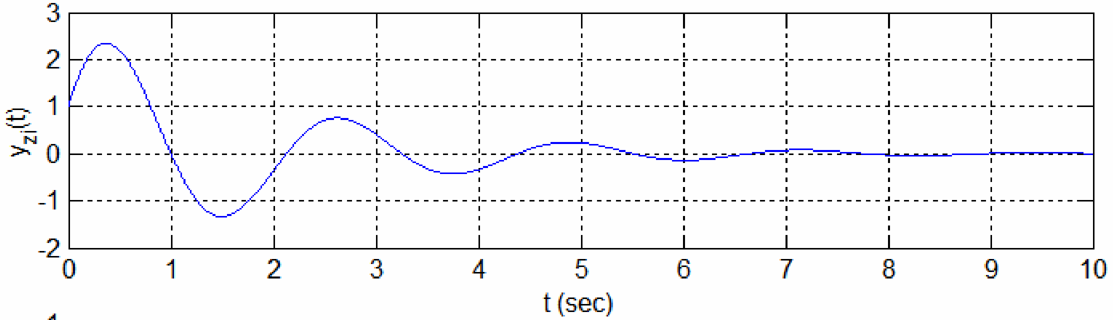
$$\beta\frac{s+\alpha}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$Y(s) = \left[\frac{s+8}{s^2+s+8}\right] + \left[\frac{0.25}{s} - 0.25\frac{s+1}{s^2+s+8}\right]$$

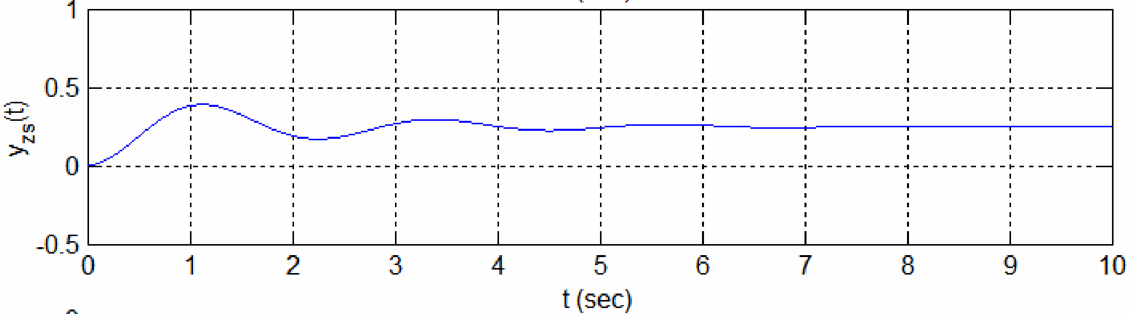
LT Table

$$y(t) = 2.87e^{-0.5t}\sin(2.78t+0.36) + 0.25 - 0.25e^{-0.5t}\sin(2.78t+1.4)$$

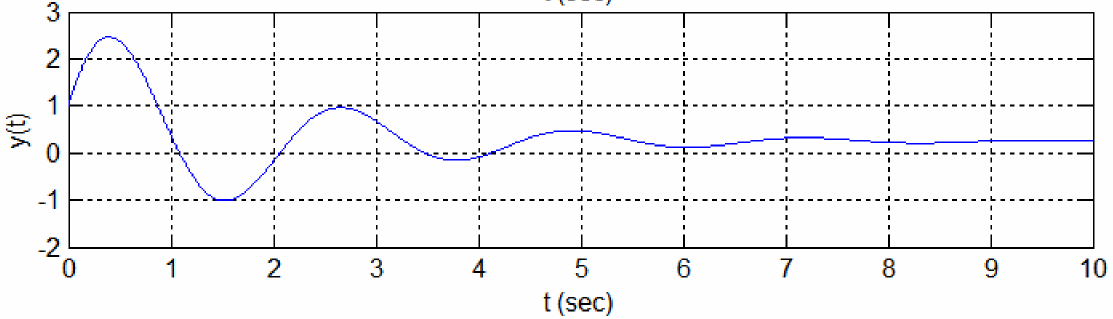
Zero-Input Response



Zero-State Response



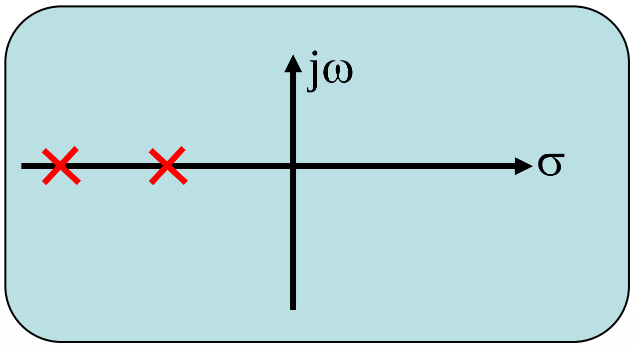
Total Response

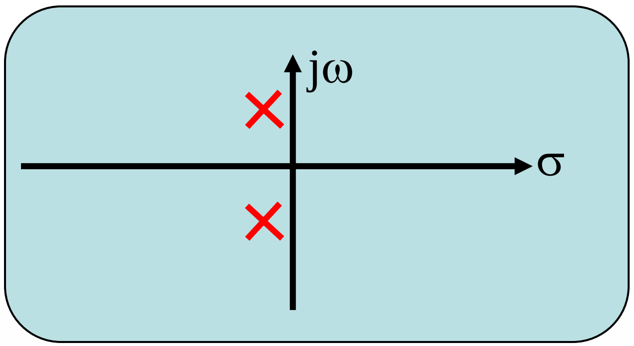


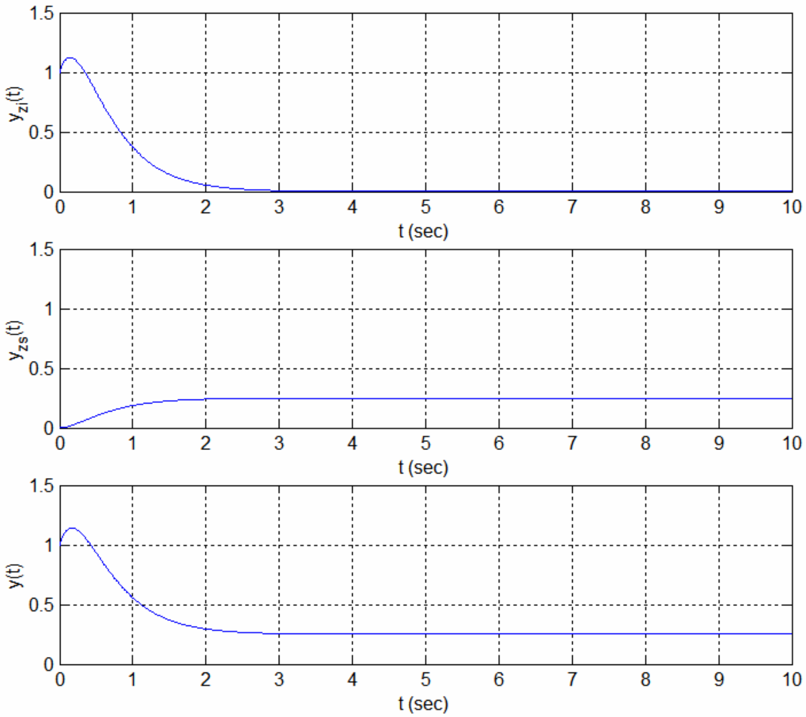
Compare the Two Cases:

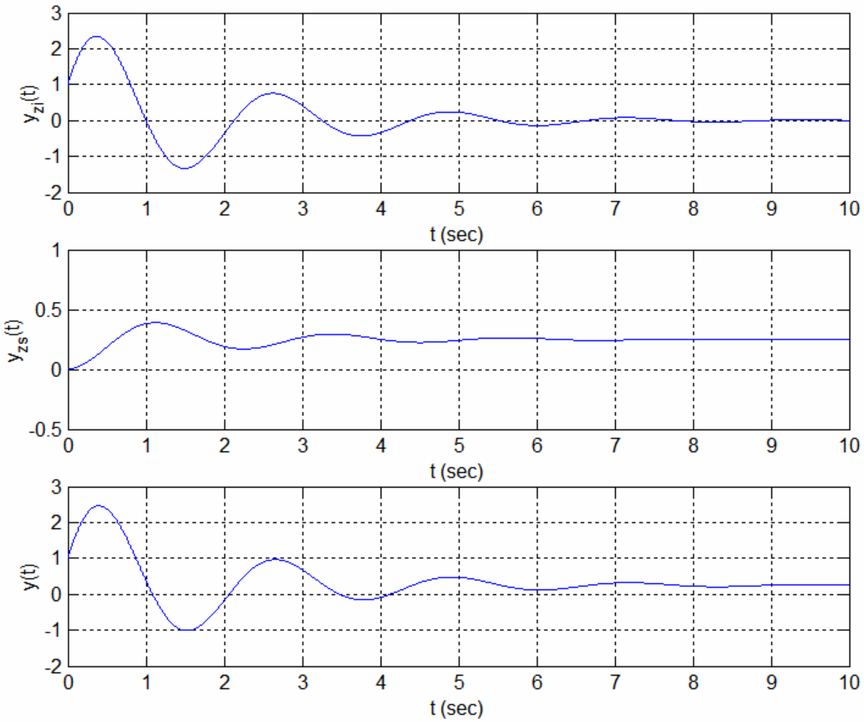
$$H(s)=\frac{2}{s^2+6s+8}=\frac{2}{(s+4)(s+2)}$$

$$H(s)=\frac{2}{s^2+s+8}=\frac{2}{(s+0.5-j2.78)(s+0.5+j2.78)}$$









Ex. 6.38 "RLCLC" Circuit

In analyzing circuits we often

-Have zero IC's

-Have no specific input signal in mind

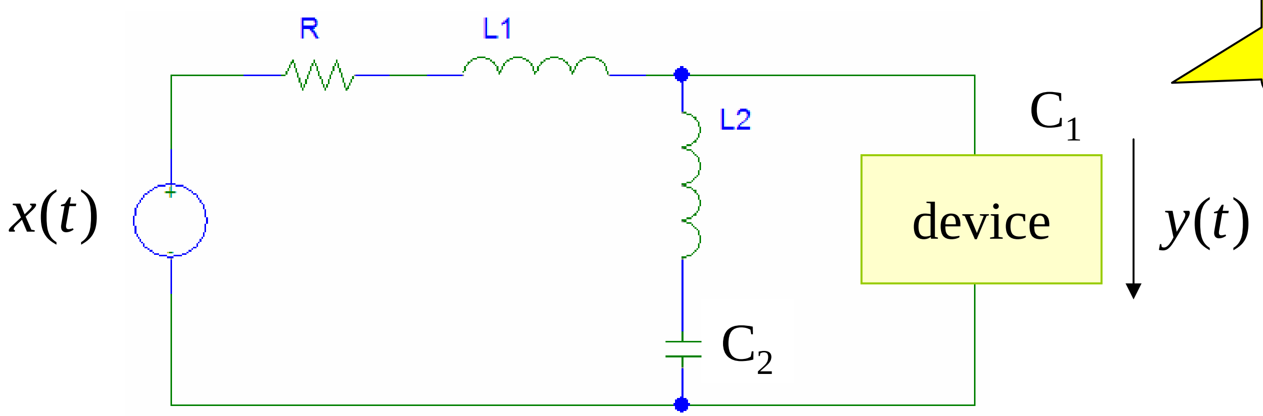
⇒ We only want to find the transfer function and/or the frequency response

Suppose you must analyze the following circuit to find the current through C_1

(You might need to do that because in the physical circuit there is a device being driven by that current and that device has an input impedance that is modeled as a capacitor)

Re-Drawn Version of Fig. 6.13… but equivalent!!

⇒



Now, replace "device" by its capacitor model and find the transfer function between:

& input = voltage $x(t)$
output = current $y(t)$

Use s-domain impedances $\begin{cases} Ls \\ 1/Cs \end{cases}$

This is the whole idea behind the LT approach!!

And analyze the circuit as if $X(s)$ is a DC voltage source.

Can find that (see book for details):

$$H(s) = \frac{C_1C_2L_2s^3 + C_1s}{C_1C_2L_1L_2s^4 + RC_1C_2L_2s^3 + \left[L_1(C_1+C_2) + L_2C_2\right]s^2 + R(C_1+C_2)s + 1}$$

Note that there are 6 "adjustable" coefficients but only 5 adjustable component values

$\Rightarrow$ Can't achieve all possible coefficient combinations

A zero "at the origin" forces low frequencies to be attenuated

A purely-imaginary-roots term… puts conjugate zeros on the *j*ω axis… nulls one specific frequency

$$H(s) = \frac{(1/L_1)s\left(s^2 + 1/C_2L_2\right)}{s^4 + (R/L_1)s^3 + \left[L_1(C_1 + C_2) + L_2C_2\right]/C_1C_2L_1L_2s^2 + R(C_1 + C_2)/C_1C_2L_1L_2s + 1/C_1C_2L_1L_2}$$

To see what this circuit can do we look at its frequency response. This is valid because all poles are in the region of convergence. (True for any passive RLC circuit)

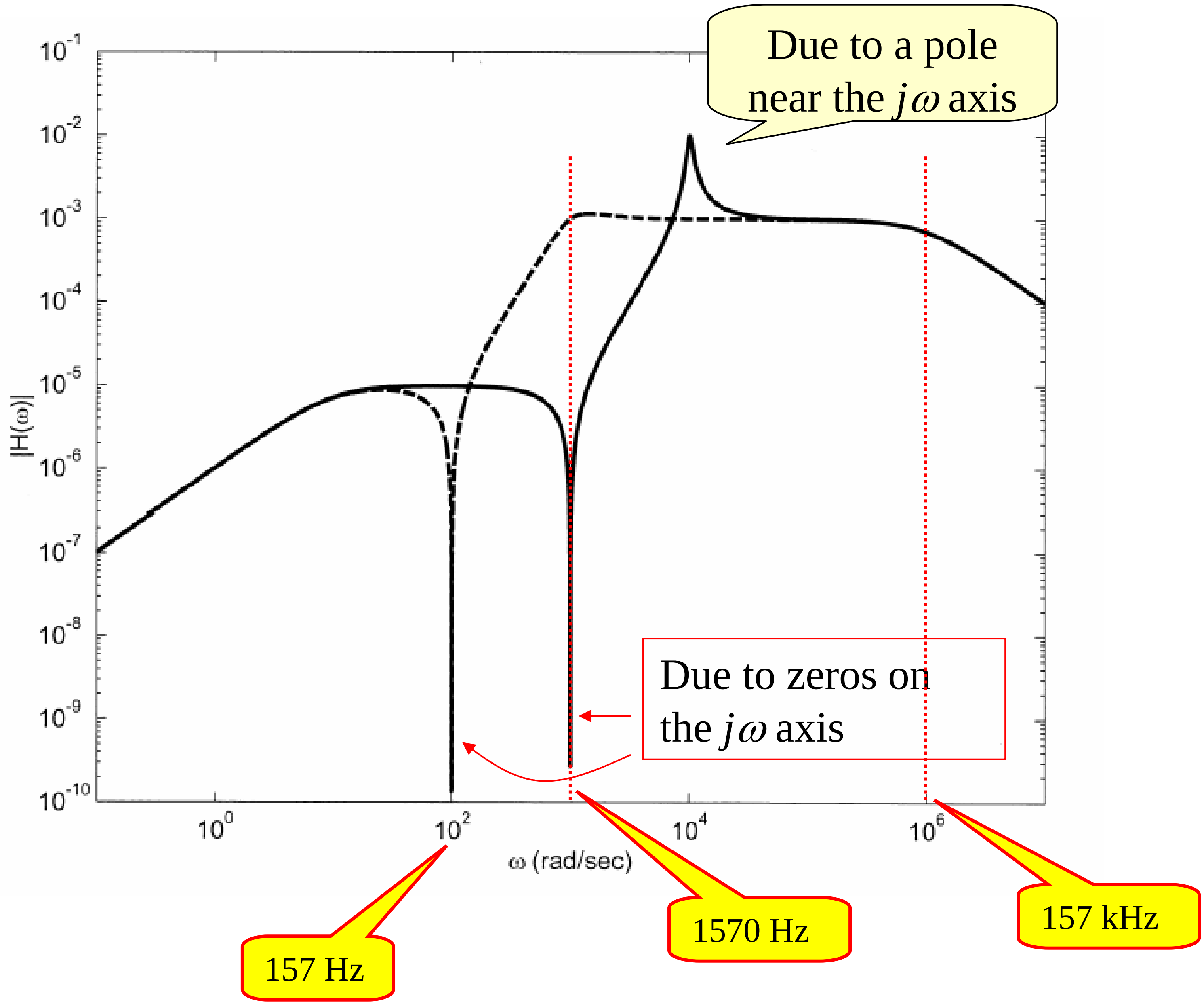
Get $H(\omega)$ by replacing $s \rightarrow j\omega$

The following plot shows $|H(\omega)|$ for two sets of component values

Note { similarities…
but differences!!

How do we choose the component values to do what we want?!

STAY TUNED!!!


Due to a pole near the $j\omega$ axis
Due to zeros on the $j\omega$ axis
157 Hz
1570 Hz
157 kHz
|H(ω)|
ω (rad/sec)
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
10^{-10}
10^{0}
10^{2}
10^{4}
10^{6}