Cramer-Rao Lower Bounds for Estimation of Phase in LBI Based Localization Systems

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Abstract—**This paper derives the Cramer-Rao lower bound (CRLB) on estimates of phase in long baseline interferometry (LBI) based localization systems. LBI localization is a classical method for finding the location of a non-cooperative emitter by estimating the phase difference between received signals by two sensors spatially separated on a single platform. In this paper, we derive the CRLB for phase difference in LBI-based systems by modelling the received signal as a deterministic unknown; that is, its samples are considered as nuisance parameters to be estimated. Consequently, the CRLB computations become much more complicated in this case. Finally, we provide the discussion for our results.**

I. Introduction

Passive localization is an important problem that has been investigated for many years [1]-[14]. There are several efficient methods to perform passive localization based on measuring one or more position-dependent parameters of the received signals such as angle of arrival (AOA), time difference of arrival (TDOA), frequency difference of arrival (FDOA) or the energy of the received signal. The classic approach used in all these methods is to first estimate these position-dependent parameters from many received signals (or in some methods, pairs of received signals) and then use the collection of estimated parameters in a second estimation stage to determine an estimate of the emitter's location using some statistical inference techniques like least-squares or maximum likelihood [1]. Thus, to evaluate the accuracy of an emitter location method, it is first necessary to evaluate the accuracy of the position-dependent parameter(s) estimation that is governed by the Cramer-Rao lower bound (CRLB) of the parameter(s).

Localization based on long baseline interferometry (LBI) is a classical method for finding the location of a non-cooperative emitter by estimating the phase difference between the received signals by two sensors that have been spatially separated on a *single platform* [4]. The LBI method was compared to a differential Doppler method in [4], although the inherent first-stage accuracies of these two methods were not set based on any CRLB analysis. Thus, the result presented here together with other CRLB results (e.g., [9]) would allow a fairer comparison between the two location methods considered in [4].

Under the so-called narrowband assumption we can write the noise-free *analytic* model of our received signals at sensor 1 and sensor 2 as

$$\begin{aligned} x_1(t) &= e^{j\omega_c t} s(t) \\ x_2(t) &= \alpha e^{j\omega_c (t-\tau)} e^{-j\omega_d t} s(t-\tau), \end{aligned} \tag{1}$$

where $s(t)$ is the low-pass equivalent signal (also called the complex envelope) of the noise-free signal at the first sensor, α is the unknown relative gain, ω_c is the carrier frequency, ω_d is the unknown Doppler difference and τ is the unknown delay between the two sensors.

In the LBI method the two sensors are quite close to each other (e.g two antennas on one aircraft). Thus, they each will have approximately the same relative velocity with respect to the emitter, therefore the Doppler difference is almost zero. Likewise, since the delay difference is so small, the effect of the delay on the complex envelope is negligible. Thus, the approximated noise-free signal model for the LBI-based method is

$$\begin{aligned} x_1(t) &= e^{j\omega_c t} s(t) \\ x_2(t) &\approx \alpha e^{j\omega_c (t-\tau)} s(t). \end{aligned}$$

In other words, since the two sensors are spatially close to each other in LBI systems, the only effect of delay happens on the carrier. We can write the *low-pass equivalent* of the received signals corrupted by noise as

$$\begin{aligned} r_1(t) &= s(t) + w_1(t) \\ r_2(t) &= \alpha e^{j\varphi} s(t) + w_2(t), \\ \varphi &\triangleq -\omega_c \tau \end{aligned} \tag{2}$$

where φ is the phase difference and $w_1(t)$ and $w_2(t)$ are complex zero-mean white Gaussian noise processes. Thus, we can say that in the LBI case, the location of the emitter shows up only as a differential phase between two received signals.

In LBI the antenna spacing is large enough to induce an ambiguity in the measured phase difference; however, as has been pointed out in [5], the least-squares location algorithm can work directly with the 2π-ambiguous measurements. Thus, for our purposes here we need only assess the accuracy of phase measurement and can ignore the effect of phase wrapping.

II. CRLB for Estimation of Phase in LBI Based System

To derive the CRLB for estimation of phase in an LBI-based system, we assume the transmitted signal is an unknown deterministic signal, a common assumption when there is no prior information on the signals. Yeredor and Angel [15] calculated the CRLB for TDOA/FDOA estimation by modelling the signal as deterministic but unknown. We also use a similar model to derive the CRLB for phase in an LBI based system.

Assuming that the received signals are sampled at the Nyquist rate to yield their discrete-time versions, (2) becomes

$$r_1[n] = s[n] + w_1[n], \qquad n = 0, 1, \ldots, N-1$$
$$r_2[n] = \alpha e^{j\varphi} s[n] + w_2[n], \qquad n = 0, 1, \ldots, N-1,$$

where $w_1[n]$ and $w_2[n]$ are complex zero-mean white Gaussian noises with variances σ_1^2 and σ_2^2. Now, we can rewrite it in vector form as

$$\begin{aligned} \boldsymbol{r}_1 &= \boldsymbol{s} + \boldsymbol{w}_1 \\ \boldsymbol{r}_2 &= \alpha e^{j\varphi}\boldsymbol{s} + \boldsymbol{w}_2, \end{aligned} \tag{3}$$

where the vector $\boldsymbol{s}$ is the noise-free signal vector at the first sensor, $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are the observation vectors, and $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ are uncorrelated complex-valued white Gaussian noise vectors:

$$\boldsymbol{s} \triangleq \left[s[0]\; s[1] \ldots s[N-1] \right]^{\mathrm{T}}$$
$$\boldsymbol{r}_1 \triangleq \left[r_1[0]\; r_1[1] \ldots r_1[N-1] \right]^{\mathrm{T}}$$
$$\boldsymbol{r}_2 \triangleq \left[r_2[0]\; r_2[1] \ldots r_2[N-1] \right]^{\mathrm{T}}$$

Since $\boldsymbol{s}$ and $\alpha e^{j\varphi}\boldsymbol{s}$ are deterministic vectors, they set the means of the received vectors but have no impact on the variance. Thus, $\boldsymbol{r} \triangleq [\boldsymbol{r}_1^T \;\; \boldsymbol{r}_2^T]^T$ is a Complex Gaussian vector with mean $\boldsymbol{\mu}$ and covariance $\boldsymbol{C}$ given by

$$\boldsymbol{\mu} \triangleq \begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{s} \\ \alpha e^{j\varphi}\boldsymbol{s} \end{bmatrix}$$
$$\boldsymbol{C} \triangleq \begin{bmatrix} \sigma_1^2 \boldsymbol{I}_N & \boldsymbol{0} \\ \boldsymbol{0} & \sigma_2^2 \boldsymbol{I}_N \end{bmatrix} \tag{4}$$

We are interested in the CRLB of the parameter φ. However, there are other unknown deterministic parameters, namely α and the complex signal vector $\boldsymbol{s}$, that get involved in the analysis and must be considered in calculations. We define the parameter vector $\boldsymbol{\theta}$ with ($2N$+2) elements containing all real-valued unknown parameters as

$$\boldsymbol{\theta} = \left[\mathrm{Re}\{\boldsymbol{s}^T\} \;\; \mathrm{Im}\{\boldsymbol{s}^T\} \;\; \alpha \;\; \varphi \right]^T_{(2N+2)\times 1} \tag{5}$$

where $\mathrm{Re}\{\boldsymbol{s}^{\mathrm{T}}\}$ and $\mathrm{Im}\{\boldsymbol{s}^{\mathrm{T}}\}$ are the real part and imaginary parts of $\boldsymbol{s}^{\mathrm{T}}$. The Fisher information matrix (FIM) for real-valued parameters estimated from a complex Gaussian vector is given in [16] as

$$\boldsymbol{FIM} = tr\left[\boldsymbol{C}^{-1}\left(\frac{\partial \boldsymbol{C}}{\partial \boldsymbol{\theta}}\right)\boldsymbol{C}^{-1}\left(\frac{\partial \boldsymbol{C}}{\partial \boldsymbol{\theta}}\right)\right] + 2\,\mathrm{Re}\left\{ \left(\frac{\partial \boldsymbol{\mu}}{\partial \boldsymbol{\theta}}\right)^H \boldsymbol{C}^{-1}\left(\frac{\partial \boldsymbol{\mu}}{\partial \boldsymbol{\theta}}\right)\right\} \tag{6}$$

Since the covariance matrix does not depend on the parameters, the first term in (6) is equal to zero. From (4), we can write $\partial\boldsymbol{\mu}/\partial\boldsymbol{\theta}$ as a $2N \times (2N+2)$ matrix structured as,

$$\frac{\partial \boldsymbol{\mu}}{\partial \boldsymbol{\theta}} = \begin{bmatrix} \boldsymbol{I}_N & j\boldsymbol{I}_N & \boldsymbol{0}_{N\times 1} & \boldsymbol{0}_{N\times 1} \\ \alpha e^{j\varphi}\boldsymbol{I}_N & j\alpha e^{j\varphi}\boldsymbol{I}_N & e^{j\varphi}\boldsymbol{s} & j\alpha e^{j\varphi}\boldsymbol{s} \end{bmatrix}_{2N\times(2N+2)} \tag{7}$$

Replacing $\partial\boldsymbol{\mu}/\partial\boldsymbol{\theta}$ from (7) into (6), gives the $\boldsymbol{FIM}$ as,

$$\boldsymbol{FIM} = 2\,\mathrm{Re}\left\{ \left(\frac{\partial \boldsymbol{\mu}}{\partial \boldsymbol{\theta}}\right)^H \boldsymbol{C}^{-1}\left(\frac{\partial \boldsymbol{\mu}}{\partial \boldsymbol{\theta}}\right)\right\} =$$

$$2\,\mathrm{Re}\left\{ \begin{bmatrix} (\frac{1}{\sigma_1^2}+\frac{\alpha^2}{\sigma_2^2})\boldsymbol{I}_N & j(\frac{1}{\sigma_1^2}+\frac{\alpha^2}{\sigma_2^2})\boldsymbol{I}_N & \frac{\alpha\boldsymbol{s}}{\sigma_2^2} & j\frac{\alpha^2\boldsymbol{s}}{\sigma_2^2} \\ -j(\frac{1}{\sigma_1^2}+\frac{\alpha^2}{\sigma_2^2})\boldsymbol{I}_N & (\frac{1}{\sigma_1^2}+\frac{\alpha^2}{\sigma_2^2})\boldsymbol{I}_N & -j\frac{\alpha\boldsymbol{s}}{\sigma_2^2} & \frac{\alpha^2\boldsymbol{s}}{\sigma_2^2} \\ \frac{\alpha\boldsymbol{s}^H}{\sigma_2^2} & j\frac{\alpha\boldsymbol{s}^H}{\sigma_2^2} & \frac{E_s}{\sigma_2^2} & j\frac{\alpha E_s}{\sigma_2^2} \\ -j\frac{\alpha^2\boldsymbol{s}^H}{\sigma_2^2} & \frac{\alpha^2\boldsymbol{s}^H}{\sigma_2^2} & -j\frac{\alpha E_s}{\sigma_2^2} & \frac{\alpha^2 E_s}{\sigma_2^2} \end{bmatrix} \right\} =$$

$$\frac{2}{\sigma_2^2}\begin{bmatrix} (\lambda+\alpha^2)\boldsymbol{I}_N & \boldsymbol{0}_{N\times N} & \alpha\boldsymbol{s}_r & -\alpha^2\boldsymbol{s}_i \\ \boldsymbol{0}_{N\times N} & (\lambda+\alpha^2)\boldsymbol{I}_N & \alpha\boldsymbol{s}_i & \alpha^2\boldsymbol{s}_r \\ \alpha\boldsymbol{s}_r^T & \alpha\boldsymbol{s}_i^T & E_s & 0 \\ -\alpha^2\boldsymbol{s}_i^T & \alpha^2\boldsymbol{s}_r^T & 0 & \alpha^2 E_s \end{bmatrix}_{(2N+2)\times(2N+2)} \tag{8}$$

where,

$$\lambda \triangleq \frac{\sigma_2^2}{\sigma_1^2}, \quad \boldsymbol{s}_r \triangleq \mathrm{Re}\{\boldsymbol{s}\}, \quad \boldsymbol{s}_i \triangleq \mathrm{Im}\{\boldsymbol{s}\}, \quad E_s = \left\|\boldsymbol{s}\right\|^2 = \boldsymbol{s}^H\boldsymbol{s}.$$

Now, we can write the matrix $\boldsymbol{FIM}$ as a block matrix with the following format:

$$\boldsymbol{FIM} \triangleq \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}$$

where,

$$\boldsymbol{A} \triangleq \frac{2}{\sigma_2^2}\begin{bmatrix} (\lambda+\alpha^2).\boldsymbol{I}_N & \boldsymbol{0}_{N\times N} \\ \boldsymbol{0}_{N\times N} & (\lambda+\alpha^2).\boldsymbol{I}_N \end{bmatrix}; \quad \boldsymbol{B} \triangleq \frac{2}{\sigma_2^2}\begin{bmatrix} \alpha\boldsymbol{s}_r & -\alpha^2\boldsymbol{s}_i \\ \alpha\boldsymbol{s}_i & \alpha^2\boldsymbol{s}_r \end{bmatrix};$$
$$\boldsymbol{C} \triangleq \frac{2}{\sigma_2^2}\begin{bmatrix} \alpha\boldsymbol{s}_r^T & \alpha\boldsymbol{s}_i^T \\ -\alpha^2\boldsymbol{s}_i^T & \alpha^2\boldsymbol{s}_r^T \end{bmatrix}; \quad \boldsymbol{D} \triangleq \frac{2}{\sigma_2^2}\begin{bmatrix} E_s & 0 \\ 0 & \alpha^2 E_s \end{bmatrix}.$$

Using the block matrix inversion formula [17] yields the CRLB matrix as,

$$\boldsymbol{CRLB} = \boldsymbol{FIM}^{-1} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}^{-1} =$$
$$\begin{bmatrix} \boldsymbol{A}^{-1} + \boldsymbol{A}^{-1}\boldsymbol{B}(\boldsymbol{D} - \boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B})^{-1}\boldsymbol{C}\boldsymbol{A}^{-1} & -\boldsymbol{A}^{-1}\boldsymbol{B}(\boldsymbol{D} - \boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B})^{-1} \\ -(\boldsymbol{D} - \boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B})^{-1}\boldsymbol{C}\boldsymbol{A}^{-1} & (\boldsymbol{D} - \boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B})^{-1} \end{bmatrix}. \quad (9)$$

Extracting the lower-right sub-matrix from (9) gives the CRLB for the parameters α and φ :

$$\boldsymbol{CRLB}_{\alpha,\varphi} = (\boldsymbol{D} - \boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B})^{-1} = \frac{\sigma_2^2(\lambda+\alpha^2)}{2\lambda\alpha^2 E_s}\begin{bmatrix} \alpha^2 & 0 \\ 0 & 1 \end{bmatrix},$$

where we have used the diagonality of the matrix defined as $\boldsymbol{A}$. Finally, extracting the lower-right element from this matrix gives the CRLB on the parameter φ :

$$CRLB_{\varphi} = \frac{\sigma_2^2(\lambda+\alpha^2)}{2\lambda\alpha^2 E_s} = \frac{\sigma_1^2}{2E_s} + \frac{\sigma_2^2}{2\alpha^2 E_s}.$$

Noting that $E_s/\sigma_1^2 = N \times SNR_1$ and $\alpha^2 E_s/\sigma_2^2 = N \times SNR_2$ gives the final form of the CRLB of interest as,

$$CRLB_{\varphi} = \frac{1}{2N} \times \left(\frac{1}{SNR_1} + \frac{1}{SNR_2}\right), \quad (10)$$

where N is the number of samples and SNR_1 and SNR_2 are signal to noise ratio for the received signals at sensor 1 and sensor 2, respectively.

Note that if either signal has zero *SNR* (in ratio not dB), then the CRLB is infinite, which makes sense because in that case we are not able to estimate the phase relative to a signal at zero SNR. Also, note that when one *SNR* is much larger than the other one then we will have $CRLB_{\varphi} \approx 1/(2N \times SNR_{\min})$ which means that even if one of the signals has very low noise, a high noise in the other signal will always hurt the estimation results. Finally, given that the two antennas are quite close it is likely that $SNR_1 = SNR_2 = SNR$ in which case the result becomes $CRLB_{\varphi} = 1/(N \times SNR)$.

III. CONCLUSION

In this paper, we considered the LBI-based location system for passive emitter localization. The localization problem based on LBI is a classical method that finds the location of the emitter by estimating the phase difference between the signals received by two sensors that are spatially separated on a single platform assuming that they are reasonably close to each other. We obtained the CRLB on estimates of the phase in the LBI based system by modelling the received signal as a deterministic unknown signal. The results show that the CRLB is proportional to the summation of *SNR* reciprocals and also the reciprocal of the number of signal samples (or equivalently the number of observations). Deriving the CRLB result without assuming that the signal is unknown (e.g. that its samples are nuisance parameters) leads to very different results – in fact it can be shown that the result is different depending on how the phase difference is embedded into the two signals. On the other hand, it also can be shown that when the signal is properly taken as unknown the CRLB result is the same regardless of how the phase difference is split between the two signals. Finally, the result shows the that CRLB on LBI phase estimation does not depend at all on the signal shape or spectrum, which means that – unlike most other methods, such as Doppler-based methods or TDOA-based methods – the expected accuracy does not depend on the type of emitter being located.

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