7.10 MLE Examples

We'll now apply the MLE theory to several examples of practical signal processing problems.

These are the same examples for which we derived the CRLB in Ch. 3

1. Range Estimation
   - sonar, radar, robotics, emitter location
2. Sinusoidal Parameter Estimation (Amp., Frequency, Phase)
   - sonar, radar, communication receivers (recall DSB Example), etc.
3. Bearing Estimation
   - sonar, radar, emitter location
4. Autoregressive Parameter Estimation
   - speech processing, econometrics

We Will Cover

See Book

Ex. 1 Range Estimation Problem

Transmit Pulse: $s(t)$ nonzero over $t \in [0, T_s]$

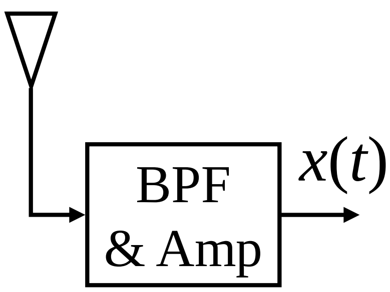
Receive Reflection: $s(t - \tau_o)$

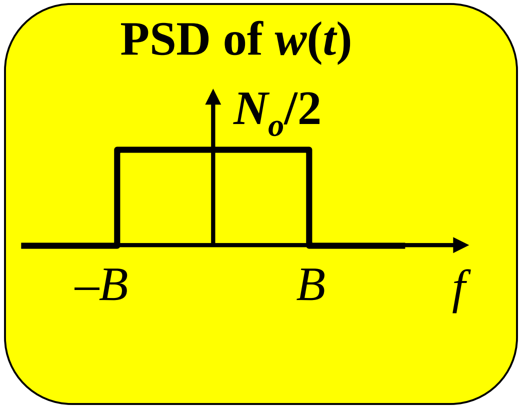
Measure Time Delay: τ_o

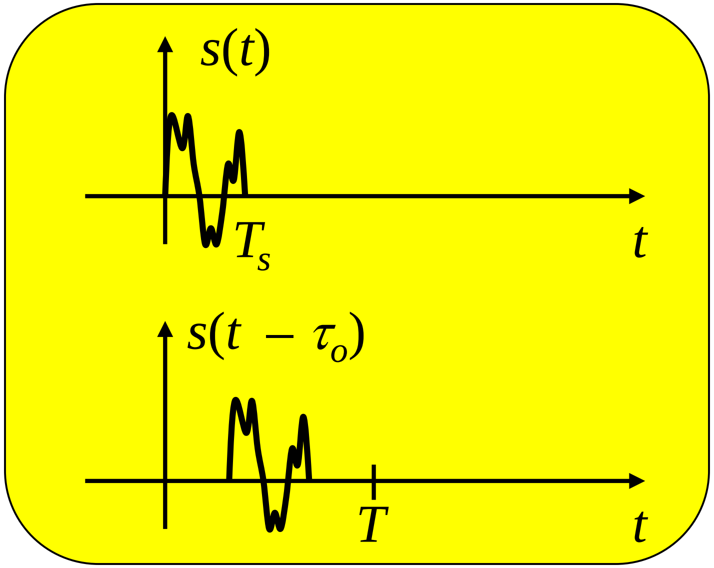
C-T Signal Model

$$x(t) = \underbrace{s(t - \tau_o)}_{s(t;\tau_o)} + w(t) \qquad 0 \le t \le T = T_s + \tau_{o,\max}$$

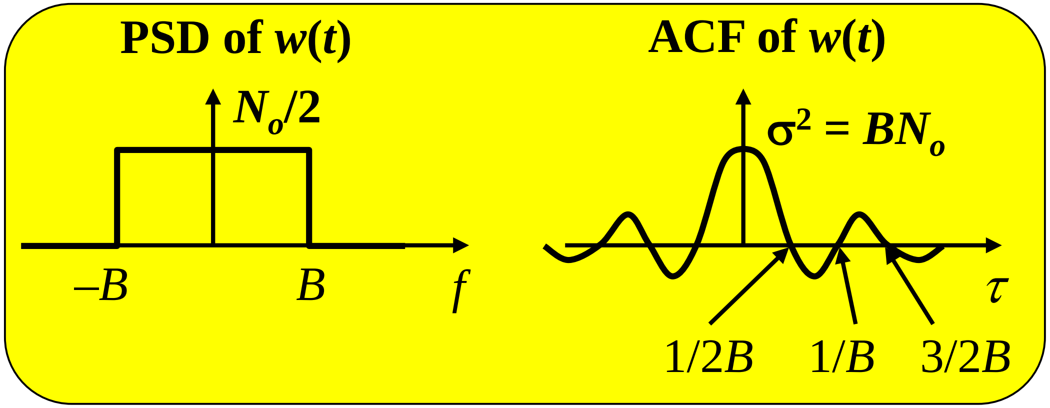
Bandlimited White Gaussian







Range Estimation D-T Signal Model



Sample Every $\Delta = 1/2B$ sec

$w[n] = w(n\Delta)$

DT White Gaussian Noise
Var $\sigma^2 = BN_o$

$$x[n] = \underbrace{s[n-n_o]}_{} + w[n] \qquad n = 0, 1, \ldots, N-1$$

$s[n;n_o]$… has M non-zero samples starting at n_o

$n_o \approx \tau_o/\Delta$

$$x[n] = \begin{cases} w[n] & 0 \le n \le n_o - 1 \\ s[n-n_o] + w[n] & n_o \le n \le n_o + M - 1 \\ w[n] & n_o + M \le n \le N - 1 \end{cases}$$

Range Estimation Likelihood Function

White and Gaussian $\Rightarrow$ Independent $\Rightarrow$ Product of PDFs

3 different PDFs – one for each subinterval

$$p(\mathbf{x};n_o) = \overbrace{\left[\prod_{n=0}^{n_o-1} C\exp\left[-\frac{x^2[n]}{2\sigma^2}\right]\right]}^{\#1} \bullet \overbrace{\left[\prod_{n=n_o}^{n_o+M-1} C\exp\left[-\frac{(x[n]-s[n-n_o])^2}{2\sigma^2}\right]\right]}^{\#2} \bullet \overbrace{\left[\prod_{n=n_o+M}^{N-1} C\exp\left[-\frac{x^2[n]}{2\sigma^2}\right]\right]}^{\#3}$$

Expand to get an $x^2[n]$ term… group it with the other $x^2[n]$ term

$$C = \frac{1}{\sqrt{2\pi\sigma^2}}$$

$$p(\mathbf{x};n_o) = \underbrace{C^N \exp\left[-\frac{\sum_{n=0}^{N-1} x^2[n]}{2\sigma^2}\right]}_{\text{Does not depend on } n_o} \bullet \exp\left[-\frac{1}{2\sigma^2}\underbrace{\sum_{n=n_o}^{n_o+M-1}(-2x[n]s[n-n_0]+s^2[n-n_o])}\right]$$

Does not depend on n_o

must minimize this or maximize its negative over values of n_o

Range Estimation ML Condition

So maximize this: $$2\sum_{n=n_o}^{n_o+M-1} x[n]s[n-n_0] + \sum_{n=n_o}^{n_o+M-1} s^2[n-n_o]$$

Because $s[n-n_o] = 0$ outside summation range… so can extend it!

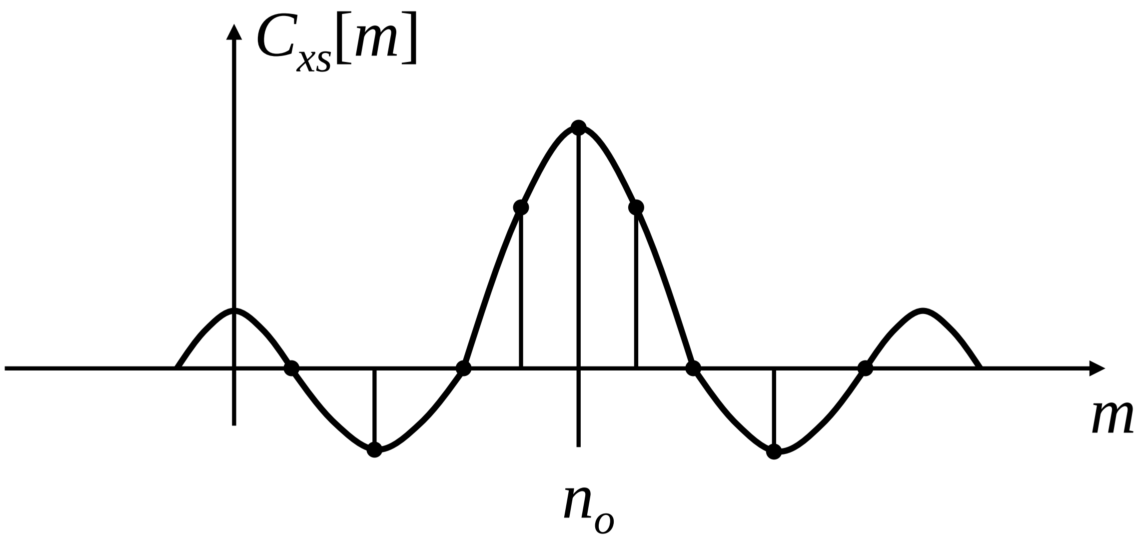
Doesn't depend on n_o! …Summand moves with the limits as n_o changes.

So maximize this: $$\sum_{n=0}^{N-1} x[n]s[n-n_0]$$

So…. MLE Implementation is based on Cross-correlation:
"Correlate" Received signal $x[n]$ with transmitted signal $s[n]$

$$\hat{n}_o = \underset{0\le m\le N-M}{\arg\max} \left\{C_{xs}[m]\right\} \qquad C_{xs}[m] = \sum_{n=0}^{N-1} x[n]s[n-m],$$

Range Estimation MLE Viewpoint



Warning: When signals are complex (e.g., ELPS) take find peak of $|C_{xs}[m]|$

$$C_{xs}[m] = \sum_{n=0}^{N-1} x[n]s[n-m],$$

- Think of this as an inner product for each m
- Compare data $x[n]$ to all possible delays of signal $s[n]$
  - pick n_o to make them most alike

Ex. 2 Sinusoid Parameter Estimation Problem

Given DT signal samples of a sinusoid in noise….

Estimate its amplitude, frequency, and phase

$$x[n] = A\cos(\Omega_o n + \phi) + w[n] \qquad n = 0, 1, \ldots, N-1$$

Ω_o is DT frequency in cycles/sample: $0 < \Omega_o < \pi$

DT White Gaussian Noise Zero Mean & Variance of σ^2

Multiple parameters… so parameter vector: $\boldsymbol{\theta} = [A \quad \Omega_o \quad \phi]^T$

The **likelihood function** is:

$$p(\mathbf{x};\boldsymbol{\theta}) = C^N \exp\left[-\frac{1}{2\sigma^2} \underbrace{\sum_{n=0}^{N-1} (x[n] - A\cos(\Omega_o n + \phi))^2}_{\triangleq J(A, \Omega_o, \phi)} \right]$$

For MLE: Minimize This

Sinusoid Parameter Estimation ML Condition

To make things easier…

Define an equivalent parameter set:

$[\alpha_1 \quad \alpha_2 \quad \Omega_o\,]^T \qquad \alpha_1 = A\cos(\phi) \quad \alpha_2 = -A\sin(\phi)$

Then… $J'(\alpha_1, \alpha_2, \Omega_o) = J(A, \Omega_o, \phi)$ $\qquad \boldsymbol{\alpha} = [\alpha_1 \quad \alpha_2]^T$

Define:

$$\mathbf{c}(\Omega_o) = [1 \quad \cos(\Omega_o) \quad \cos(\Omega_o 2) \quad \ldots \quad \cos(\Omega_o(N\text{-}1))]^T$$

$$\mathbf{s}(\Omega_o) = [0 \quad \sin(\Omega_o) \quad \sin(\Omega_o 2) \quad \ldots \quad \sin(\Omega_o(N\text{-}1))]^T$$

and…

$$\mathbf{H}(\Omega_o) = [\mathbf{c}(\Omega_o) \quad \mathbf{s}(\Omega_o)] \qquad \text{an } N\text{x2 matrix}$$

Then: $J'(\alpha_1, \alpha_2, \Omega_o) = [\mathbf{x} - \mathbf{H}(\Omega_o)\boldsymbol{\alpha}]^T [\mathbf{x} - \mathbf{H}(\Omega_o)\boldsymbol{\alpha}]$

Looks like the linear model case… except for Ω_o dependence of $\mathbf{H}(\Omega_o)$

Thus, for any fixed Ω_o value, the optimal $\boldsymbol{\alpha}$ estimate is

$$\hat{\boldsymbol{\alpha}} = \left[\mathbf{H}^T(\Omega_o)\mathbf{H}(\Omega_o)\right]^{-1}\mathbf{H}^T(\Omega_o)\mathbf{x}$$

Then plug that into $J'(\alpha_1, \alpha_2, \Omega_o)$:

$$\begin{aligned}
J'(\hat{\alpha}_1, \hat{\alpha}_2, \Omega_o) &= \left[\mathbf{x} - \mathbf{H}(\Omega_o)\hat{\boldsymbol{\alpha}}\right]^T \left[\mathbf{x} - \mathbf{H}(\Omega_o)\hat{\boldsymbol{\alpha}}\right] \\
&= \left[\mathbf{x}^T - \hat{\boldsymbol{\alpha}}^T\mathbf{H}^T(\Omega_o)\right]\left[\mathbf{x} - \mathbf{H}(\Omega_o)\hat{\boldsymbol{\alpha}}\right] \\
&= \mathbf{x}^T \underbrace{\left[\mathbf{I} - \mathbf{H}(\Omega_o)\left[\mathbf{H}^T(\Omega_o)\mathbf{H}(\Omega_o)\right]^{-1}\mathbf{H}^T(\Omega_o)\right]^2}_{=\mathbf{I} - \mathbf{H}(\Omega_o)\left[\mathbf{H}^T(\Omega_o)\mathbf{H}(\Omega_o)\right]^{-1}\mathbf{H}^T(\Omega_o)} \mathbf{x} \\
&= \mathbf{x}^T\mathbf{x} - \underbrace{\mathbf{x}^T\mathbf{H}(\Omega_o)\left[\mathbf{H}^T(\Omega_o)\mathbf{H}(\Omega_o)\right]^{-1}\mathbf{H}^T(\Omega_o)\mathbf{x}}_{\text{maximize w.r.t. } \Omega_o}
\end{aligned}$$

Sinusoid Parms. Exact MLE Procedure

Step 1: Maximize "this term" over Ω_o to find $\hat{\Omega}_o$

$$\hat{\Omega}_o = \arg\max_{0<\Omega_o<\pi} \left\{ \mathbf{x}^T \mathbf{H}(\Omega_o) \left[\mathbf{H}^T(\Omega_o)\mathbf{H}(\Omega_o) \right]^{-1} \mathbf{H}^T(\Omega_o)\mathbf{x} \right\}$$

Could Do Numerically

Step 2: Use result of Step 1 to get

$$\hat{\boldsymbol{\alpha}} = \left[\mathbf{H}^T(\hat{\Omega}_o)\mathbf{H}(\hat{\Omega}_o) \right]^{-1} \mathbf{H}^T(\hat{\Omega}_o)\mathbf{x}$$

Step 3: Convert Step 2 result by solving

$$\hat{\alpha}_1 = \hat{A}\cos(\hat{\phi})$$

$$\hat{\alpha}_2 = -\hat{A}\sin(\hat{\phi})$$

for $\hat{A}$ & $\hat{\phi}$

Sinusoid Parms. Approx. MLE Procedure

First we look at a specific structure:

$$\mathbf{x}^T\mathbf{H}(\Omega_o)\left[\mathbf{H}^T(\Omega_o)\mathbf{H}(\Omega_o)\right]^{-1}\mathbf{H}^T(\Omega_o)\mathbf{x} = \begin{bmatrix}\mathbf{c}^T(\Omega_o)\mathbf{x}\\ \mathbf{s}^T(\Omega_o)\mathbf{x}\end{bmatrix}^T \underbrace{\begin{bmatrix}\mathbf{c}^T(\Omega_o)\mathbf{c}(\Omega_o) & \mathbf{c}^T(\Omega_o)\mathbf{s}(\Omega_o)\\ \mathbf{s}^T(\Omega_o)\mathbf{c}(\Omega_o) & \mathbf{s}^T(\Omega_o)\mathbf{s}(\Omega_o)\end{bmatrix}^{-1}} \begin{bmatrix}\mathbf{c}^T(\Omega_o)\mathbf{x}\\ \mathbf{s}^T(\Omega_o)\mathbf{x}\end{bmatrix}$$

Then… if Ω_o is not near 0 or π, then approximately

$$\approx \begin{bmatrix}\frac{N}{2} & 0\\ 0 & \frac{N}{2}\end{bmatrix}^{-1}$$

and Step 1 becomes

$$\hat{\Omega}_o = \arg\max_{0<\Omega_o<\pi}\left\{\frac{2}{N}\left|\sum_{n=0}^{N-1}x[n]e^{-j\Omega_o n}\right|^2\right\} = \arg\max_{0<\Omega<\pi}\left\{\left|X(\Omega)\right|^2\right\}$$

DTFT of Data $x[n]$

and Steps 2 & 3 become

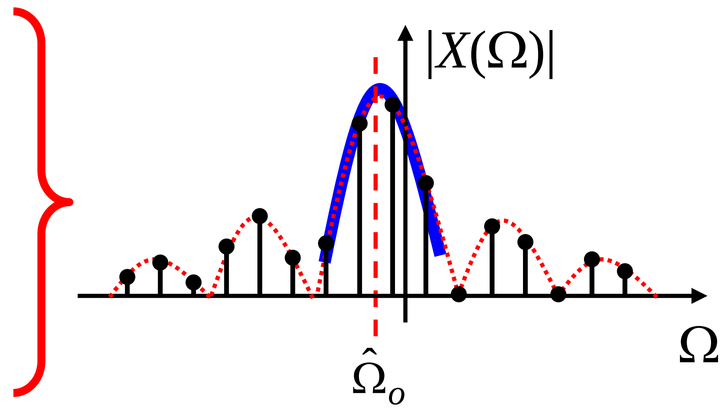
$$\hat{A} = \frac{2}{N}\left|X(\hat{\Omega}_o)\right|$$

$$\hat{\phi} = \angle X(\hat{\Omega}_o)$$

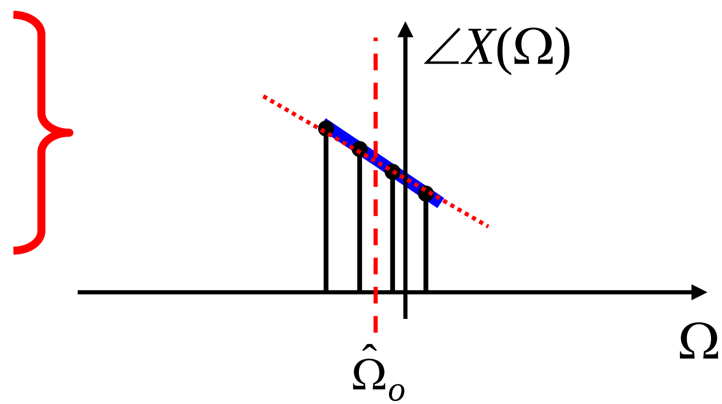
The processing is implemented as follows:

Given the data: $x[n], \quad n = 0, 1, 2, \ldots, N\text{-}1$

1. Compute the DFT $X[m], \quad m = 0, 1, 2, \ldots, M\text{-}1$ of the data
    - Zero-pad to length $M = 4N$ to ensure <u>dense grid</u> of frequency points
    - Use the FFT algorithm for computational efficiency
2. Find location of peak
    - Use quadratic interpolation of $|X[m]|$
3. Find height at peak
    - Use quadratic interpolation of $|X[m]|$

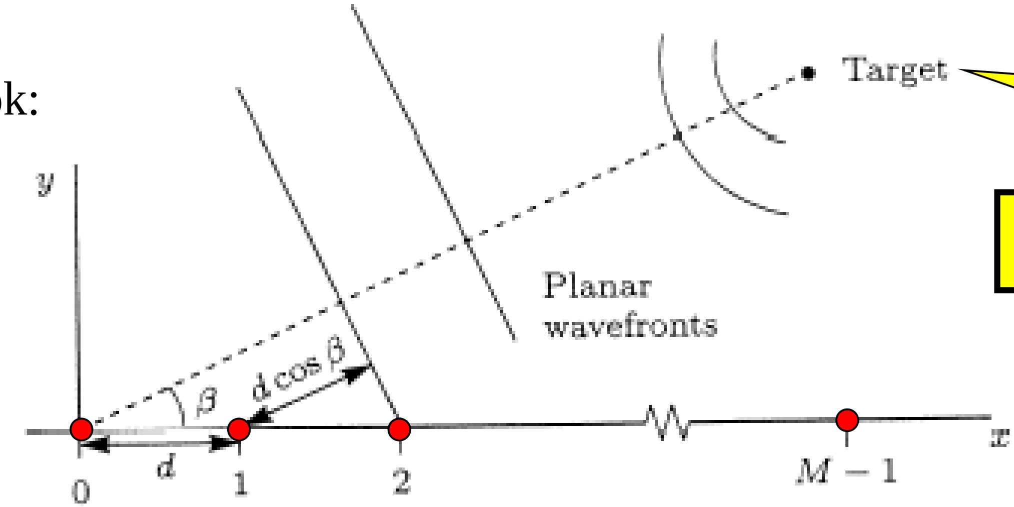


4. Find angle at peak
    - Use linear interpolation of $\angle X[m]$



Ex. 3 Bearing Estimation MLE

Figure 3.8
from textbook:



Emits or reflects signal $s(t)$

$$s(t) = A_t \cos(2\pi f_o t + \phi)$$

Simple model

Grab one "snapshot" of all M sensors at a single instant t_s:

$$x[n] = s_n(t_s) + w[n] = A\cos\left(\Omega_s n + \tilde{\phi}\right) + w[n]$$

Same as Sinusoidal Estimation!!
So… Compute DFT and Find Location of Peak!!

If emitted signal is not a sinusoid… then you get a different MLE!!