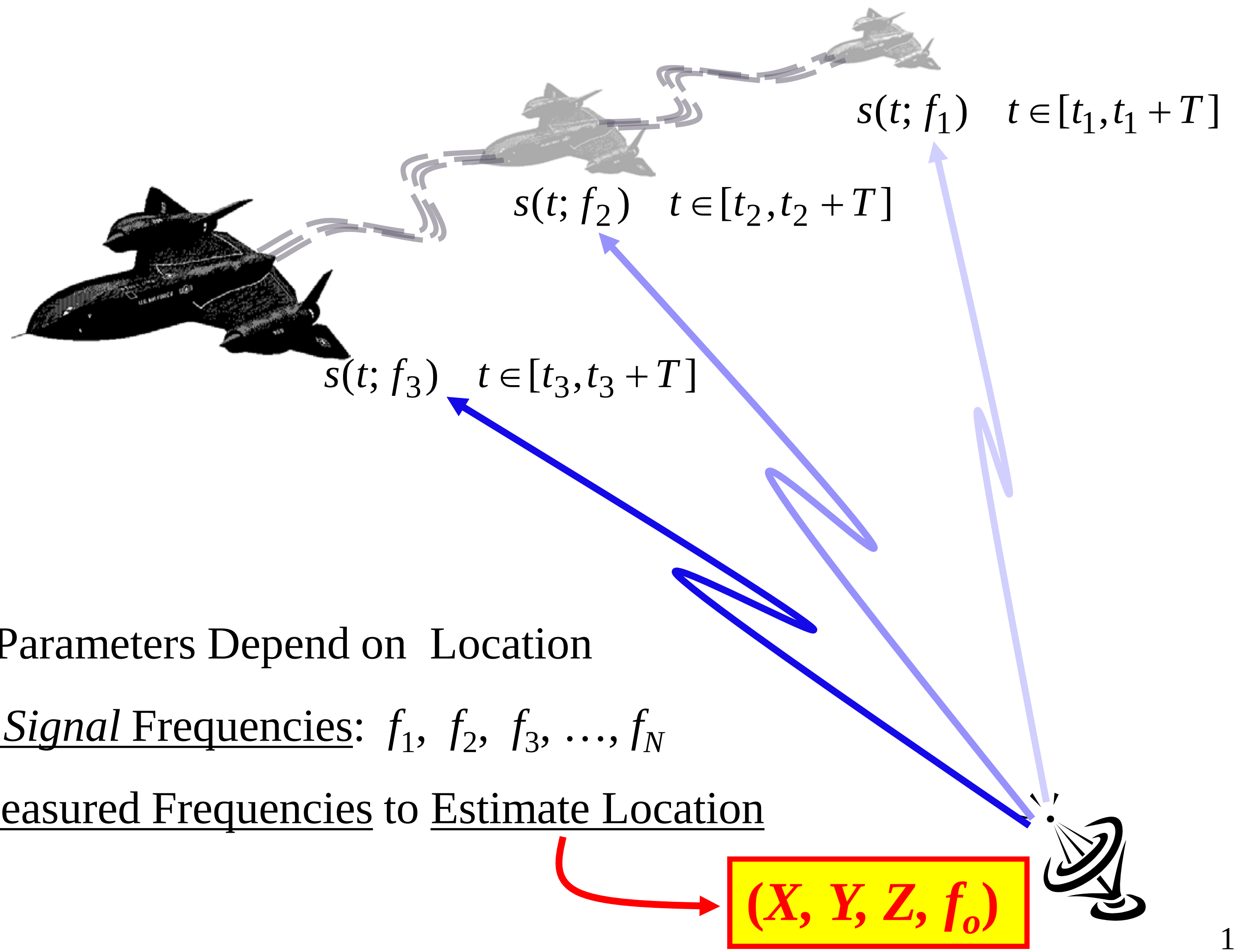
CRLB Example: Single-Rx Emitter Location via Doppler



- Received Signal Parameters Depend on Location
  - Estimate *Rx Signal* Frequencies: $f_1, f_2, f_3, \ldots, f_N$
  - Then Use Measured Frequencies to Estimate Location

Problem Background

Radar to be Located: at Unknown Location (X,Y,Z)
Transmits Radar Signal at Unknown Carrier Frequency f_o

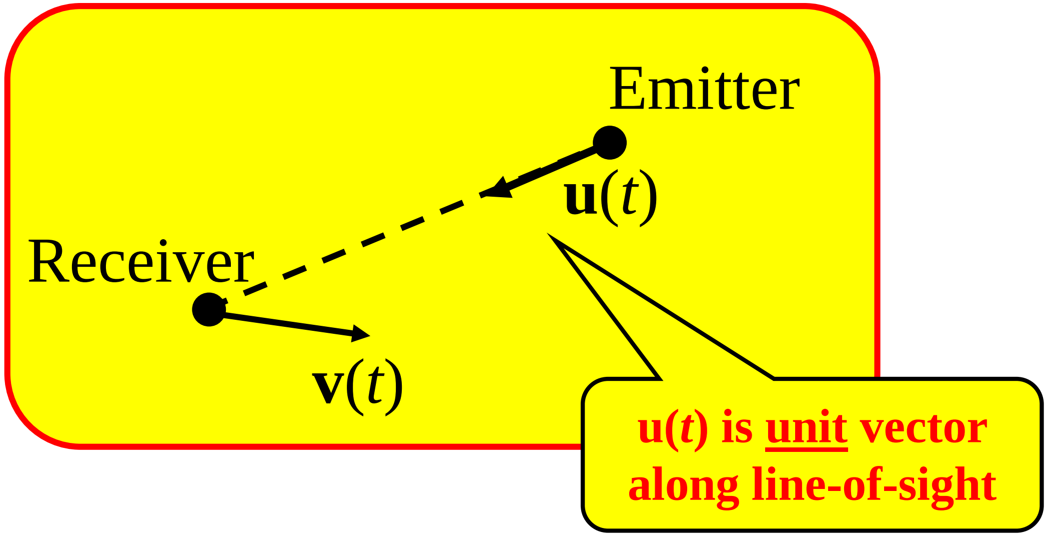
Signal is intercepted by airborne receiver with:

Known (Navigation Data):

Antenna Positions: $(X_p(t), Y_p(t), Z_p(t))$
Antenna Velocities: $(V_x(t), V_y(t), V_z(t))$

Goal: Estimate Parameter Vector $\mathbf{x} = [X\ Y\ Z\ \ f_o]^T$

Physics of Problem



Relative motion between emitter and receiver causes a Doppler shift of the carrier frequency:

$$f(t,\mathbf{x}) = f_o - \frac{f_o}{c}\mathbf{v}(t)\bullet\mathbf{u}(t)$$

$$= f_o - \frac{f_o}{c}\left[\frac{V_x(t)\left(X_p(t)-X\right)+V_y(t)\left(Y_p(t)-Y\right)+V_z(t)\left(Z_p(t)-Z\right)}{\sqrt{\left(X_p(t)-X\right)^2+\left(Y_p(t)-Y\right)^2+\left(Z_p(t)-Z\right)^2}}\right].$$

Because we estimate the frequency there is an error added:

$$\tilde{f}(t_i,\mathbf{x}) = f(t_i,\mathbf{x}) + v(t_i)$$

Estimation Problem Statement

Vector-Valued function of a Vector

Given:

Data Vector: $\tilde{\mathbf{f}}(\mathbf{x}) = \left[\tilde{f}(t_1, \mathbf{x})\ \tilde{f}(t_2, \mathbf{x}) \cdots \tilde{f}(t_N, \mathbf{x})\right]^T$

Navigation Info:
$$X_p(t_1), X_p(t_2), \cdots, X_p(t_N)$$
$$Y_p(t_1), Y_p(t_2), \cdots, Y_p(t_N)$$
$$Z_p(t_1), Z_p(t_2), \cdots, Z_p(t_N)$$
$$V_x(t_1), V_x(t_2), \cdots, V_x(t_N)$$
$$V_y(t_1), V_y(t_2), \cdots, V_y(t_N)$$
$$V_z(t_1), V_z(t_2), \cdots, V_z(t_N)$$

Estimate:

Parameter Vector: $[X\ Y\ Z\ f_o]^{\mathrm{T}}$

Right now only want to consider the CRLB

The CRLB

Note that this is a "signal" plus noise scenario:

- The "signal" is the noise-free frequency values
- The "noise" is the error made in measuring frequency

Assume zero-mean Gaussian noise with covariance matrix **C:**

- Can use the "General Gaussian Case" of the CRLB
- Of course validity of this depends on how closely the errors of the frequency estimator really do follow this

Our data vector is distributed according to: $\tilde{\mathbf{f}}(\mathbf{x}) \sim \mathcal{N}(\mathbf{f}(\mathbf{x}), \mathbf{C})$

Only Mean Shows Dependence on parameter **x**!

Only need the first term in the CRLB equation:

$$[\mathbf{J}]_{ij} = \left[\frac{\partial \mathbf{f}(\mathbf{x})}{\partial x_i}\right]^T \mathbf{C}^{-1} \left[\frac{\partial \mathbf{f}(\mathbf{x})}{\partial x_j}\right]$$

I use **J** for the FIM instead of **I** to avoid confusion with the identity matrix.

Convenient Form for FIM

Called "The Jacobian" of **f**(**x**)

To put this into an easier form to look at… Define a matrix **H**:

$$\mathbf{H} = \frac{\partial}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}) \Big|_{\mathbf{x}=\text{true value}} = \left[\mathbf{h}_1 \mid \mathbf{h}_2 \mid \mathbf{h}_3 \mid \mathbf{h}_4 \right]$$

where

$$\mathbf{h}_j = \begin{bmatrix} \frac{\partial}{\partial x_j} f(t_1, \mathbf{x}) \\ \frac{\partial}{\partial x_j} f(t_2, \mathbf{x}) \\ \vdots \\ \frac{\partial}{\partial x_j} f(t_N, \mathbf{x}) \end{bmatrix} \Bigg|_{\mathbf{x}=\text{true value}}$$

Then it is east to verify that the FIM becomes:

$$\mathbf{J} = \mathbf{H}^T \mathbf{C}^{-1} \mathbf{H}$$

CRLB Matrix

The Cramer-Rao bound covariance matrix then is:

$$\mathbf{C}_{CRB}(\mathbf{x}) = \mathbf{J}^{-1}$$
$$= \left[\mathbf{H}^T \mathbf{C}^{-1} \mathbf{H}\right]^{-1}$$

A closed-form expression for the partial derivatives needed for **H** can be computed in terms of an arbitrary set of navigation data – see "Reading Version" of these notes (on BlackBoard).

Given: Emitter Location & Platform Trajectory and Measurement Cov **C**

- Compute Matrix **H**
- Compute the CRLB covariance matrix $\mathbf{C}_{\mathrm{CRB}}(\mathbf{x})$
- Compute eigen-analysis of $\mathbf{C}_{\mathrm{CRB}}(\mathbf{x})$
- Determine the 4-D error ellipsoid.

Can't really plot a 4-D ellipsoid!!!

But… it *is* possible to project this 4-D ellipsoid down into a 2-D ellipse so that you can see the effect of geometry.

Projection of Error Ellipsoids

See also "Slice Of Error Ellipsoids"

A zero-mean Gaussian vector of two vectors **x** & **y**: $\boldsymbol{\theta} = \begin{bmatrix} \mathbf{x}^T & \mathbf{y}^T \end{bmatrix}^T$

The the PDF is:

$$p(\boldsymbol{\theta}) = \frac{1}{(2\pi)^{N/2}\sqrt{\det(\mathbf{C}_{\boldsymbol{\theta}})}} \exp\left\{-\tfrac{1}{2}\boldsymbol{\theta}^T \mathbf{C}_{\boldsymbol{\theta}}^{-1}\boldsymbol{\theta}\right\} \qquad \mathbf{C}_{\boldsymbol{\theta}} = \begin{bmatrix} \mathbf{C}_{\mathbf{x}} & \mathbf{C}_{\mathbf{xy}} \\ \mathbf{C}_{\mathbf{yx}} & \mathbf{C}_{\mathbf{y}} \end{bmatrix}$$

The quadratic form in the exponential defines an ellipse:

$$\boldsymbol{\theta}^T \mathbf{C}_{\boldsymbol{\theta}}^{-1}\boldsymbol{\theta} = k$$

Can choose k to make size of ellipsoid such that $\boldsymbol{\theta}$ falls inside the ellipsoid with a desired probability

Q: If we are given the covariance $\mathbf{C}_{\boldsymbol{\theta}}$ how is **x** alone is distributed?

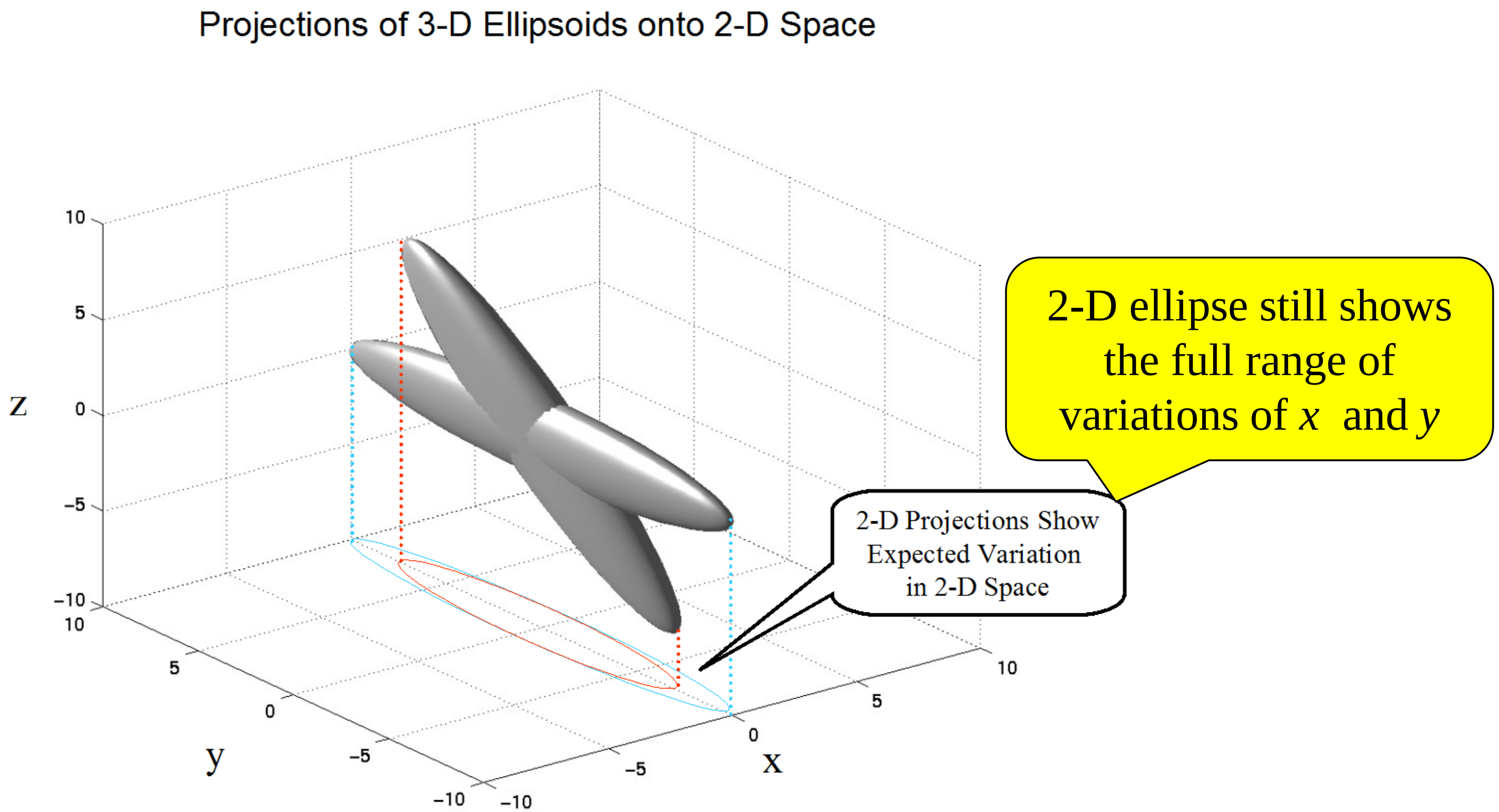
A: Extract the sub-matrix $\mathbf{C}_{\mathbf{x}}$ out of $\mathbf{C}_{\boldsymbol{\theta}}$

Projection Example

Full Vector: $\boldsymbol{\theta} = [x \ y \ z]^T$ Sub-Vector: $\mathbf{x} = [x \ y]^T$

We want to project the 3-D ellipsoid for $\boldsymbol{\theta}$

…down into a 2-D ellipse for $\mathbf{x}$



Finding Projections

To find the projection of the CRLB ellipse:

1. Invert the FIM to get $\mathbf{C}_{\text{CRB}}$
2. Select the submatrix $\mathbf{C}_{\text{CRB,sub}}$ from $\mathbf{C}_{\text{CRB}}$
3. Invert $\mathbf{C}_{\text{CRB,sub}}$ to get $\mathbf{J}_{proj}$
4. Compute the ellipse for the quadratic form of $\mathbf{J}_{proj}$

Mathematically:

$$\mathbf{C}_{CRB,sub} = \mathbf{P}\mathbf{C}_{CRB}\mathbf{P}^T = \mathbf{P}\mathbf{J}^{-1}\mathbf{P}^T \quad \Rightarrow \quad \mathbf{J}_{proj} = \left(\mathbf{P}\mathbf{J}^{-1}\mathbf{P}^T\right)^{-1}$$

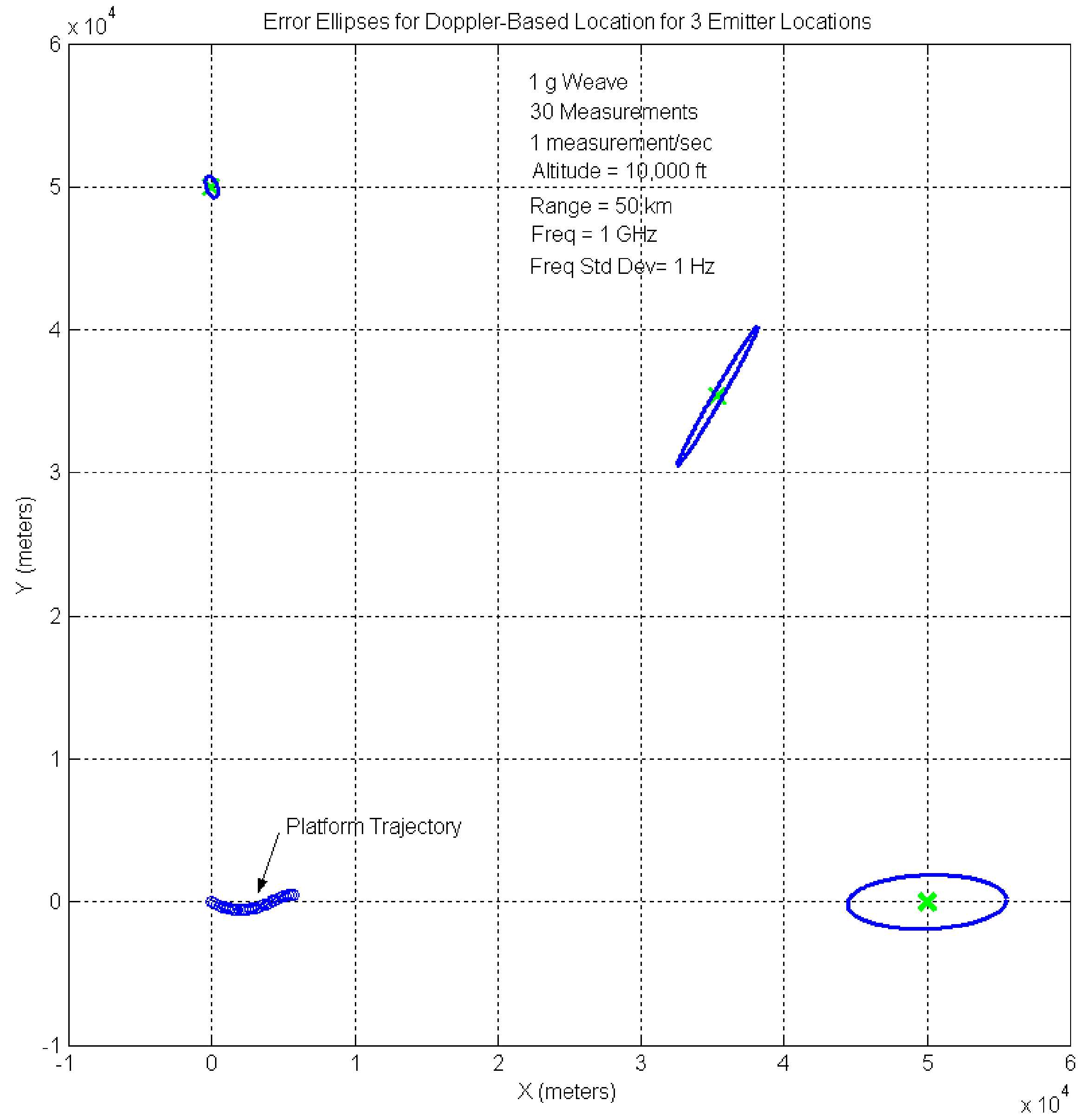
P is a matrix formed from the identity matrix: keep only the rows of the variables projecting onto

For this example, frequency-based emitter location: $[X\ Y\ Z\ f_o]^{\mathrm{T}}$

To project this 4-D error ellipsoid onto the X-Y plane:

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Projections Applied to Emitter Location



Shows 2-D ellipses that result from projecting 4-D ellipsoids

Slices of Error Ellipsoids

Q: What happens **if one parameter were perfectly known**.

Capture by setting that parameter's error to zero

$\Rightarrow$ **slice through the error ellipsoid**.

Impact:

- slice = projection when ellipsoid not tilted
- slice < projection when ellipsoid is tilted.

Recall: Correlation causes tilt