Review of Probability

Random Variable

- **Definition**

  Numerical characterization of outcome of a random event

- **Examples**

  1) Number on rolled dice
  2) Temperature at specified time of day
  3) Stock Market at close
  4) Height of wheel going over a rocky road

Random Variable

- **Non-examples**
  1) ‘Heads’ or ‘Tails’ on coin
  2) Red or Black ball from urn

But we can make these into RV’s

- **Basic Idea** – don’t know how to completely determine what value will occur
  - Can only specify probabilities of RV values occurring.

Two Types of Random Variables

Random Variable

Discrete RV

- Die
- Stocks

Continuous RV

- Temperature
- Wheel height

PDF for Continuous RV

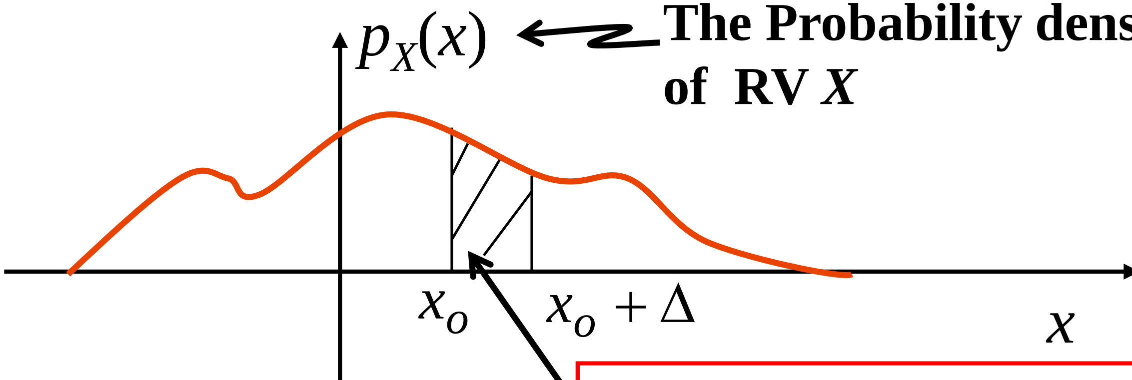
Given Continuous RV X…

What is the probability that $X = x_0$?

➡ Oddity : $P(X = x_0) = 0$

Otherwise the Prob. “Sums” to infinity

➡ Need to think of Prob. *Density* Function (PDF)



The Probability density function of RV *X*

$$P(x_0 < X < x_0 + \Delta) = \text{ area shown}$$

$$= \int_{x_o}^{x_o + \Delta} p_X(x)dx$$

Most Commonly Used PDF: Gaussian

A RV *X* with the following PDF is called a Gaussian RV

$$p_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2/2\sigma^2}$$

m & σ are parameters of the Gaussian pdf

m **= Mean of RV *X***

$\boldsymbol{\sigma}$ **= Standard Deviation of RV *X* (Note: $\boldsymbol{\sigma > 0}$)**

$\boldsymbol{\sigma^2}$ **= Variance of RV *X***

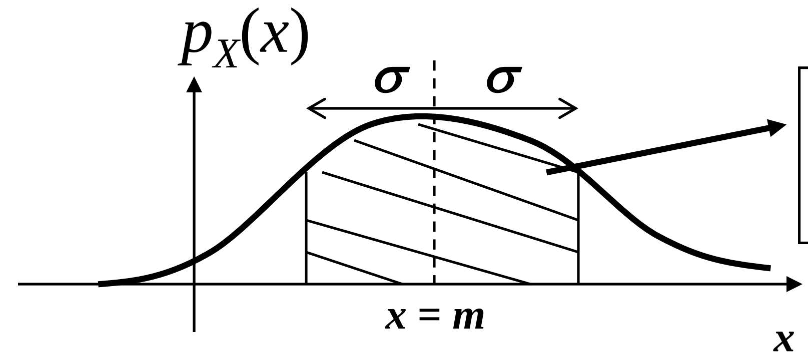
Notation: When X has Gaussian PDF we say $X \sim N(m, \sigma^2)$

Zero-Mean Gaussian PDF

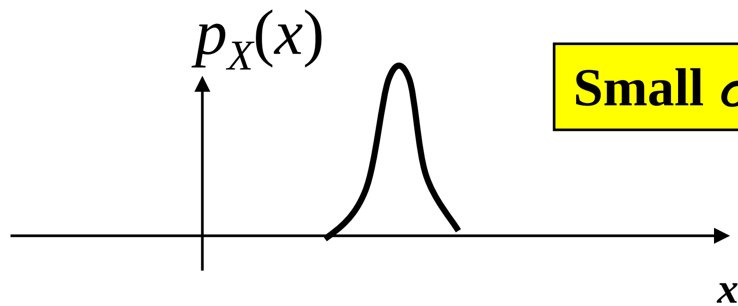
- **Generally: take the noise to be Zero Mean**

$$p_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{x^2/2\sigma^2}$$

Effect of Variance on Gaussian PDF

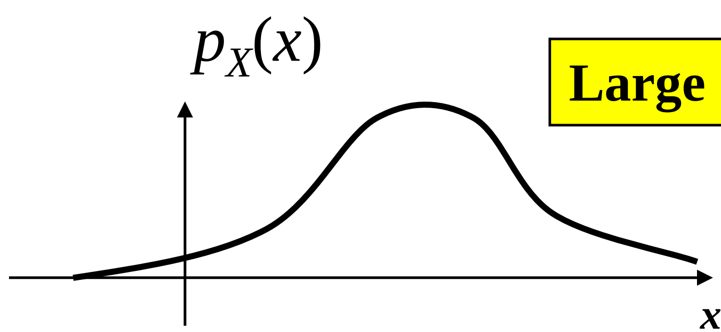


Area within ±1 σ of mean = 0.683
= 68.3%



Small σ

Small Variability
(Small Uncertainty)



Large σ

Large Variability
(Large Uncertainty)

Why Is Gaussian Used?

▪ **Central Limit theorem (CLT)**

The sum of N independent RVs has a pdf that tends to be Gaussian as $N \to \infty$

▪ **So What! Here is what : Electronic systems generate internal noise due to random motion of electrons in electronic components. The noise is the result of summing the random effects of lots of electrons.**

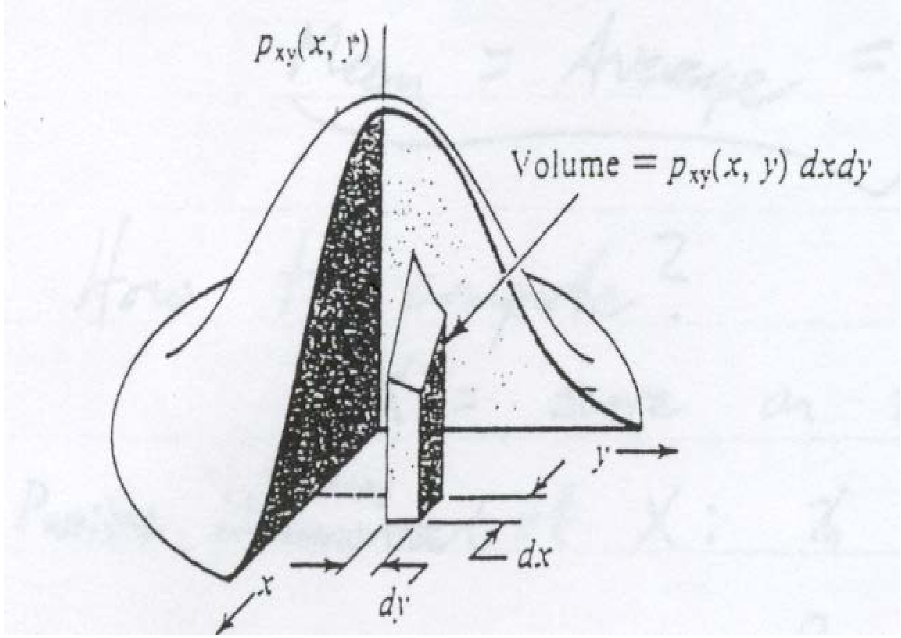
⇒ **CLT applies** ⇒ **Guassian Noise**

Joint PDF of RVs *X* and *Y*

$p_{XY}(x, y)$

Describes probabilities of joint events concerning *X* and *Y*. For example, the probability that *X* lies in interval [*a*,*b*] and *Y* lies in interval [*a*,*b*] is given by:

$$\Pr\{(a < X < b) \text{ and } (c < Y < d)\} = \int_a^b \int_c^d p_{XY}(x, y) dx dy$$



This graph shows a Joint PDF

Graph from B. P. Lathi's book: *Modern Digital & Analog Communication Systems*

Conditional PDF of Two RVs

When you have two RVs… often ask: What is the PDF of Y if X is constrained to take on a specific value.

In other words: What is the PDF of Y conditioned on the fact X is constrained to take on a specific value.

Ex.: Husband's salary X conditioned on wife's salary = $100K?

First find all wives who make EXACTLY $100K… how are their husband's salaries distributed.

Depends on the joint PDF because there are two RVs… but it should only depend on the slice of the joint PDF at Y=$100K.

Now… we have to adjust this to account for the fact that the joint PDF (even its slice) reflects how likely it is that Y=$100K will occur (e.g., if $Y=10^5$ is unlikely then $p_{XY}(x,10^5)$ will be small); so… if we divide by $p_Y(10^5)$ we adjust for this.

Conditional PDF (cont.)

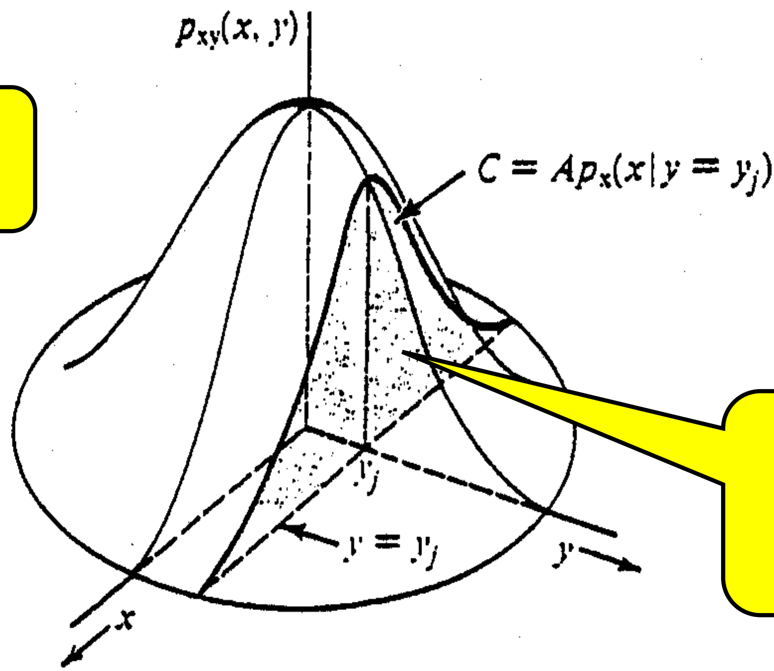
Thus, the conditional PDFs are defined as ("slice and normalize"):

$$p_{Y|X}(y\,|\,x) = \begin{cases} \dfrac{p_{XY}(x,y)}{p_X(x)}, & p_X(x) \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$p_{X|Y}(x\,|\,y) = \begin{cases} \dfrac{p_{XY}(x,y)}{p_Y(y)}, & p_Y(y) \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

x is held fixed

y is held fixed



"*slice and normalize*"
y is held fixed

This graph shows a <u>Conditional PDF</u>

Graph from B. P. Lathi's book: *Modern Digital & Analog Communication Systems*

<u>Independent RV's</u>

Independence should be thought of as saying that:

Neither RV impacts the other statistically – thus, the values that one <u>*will* likely take</u> should be irrelevant to the value that the other <u>*has* taken</u>.

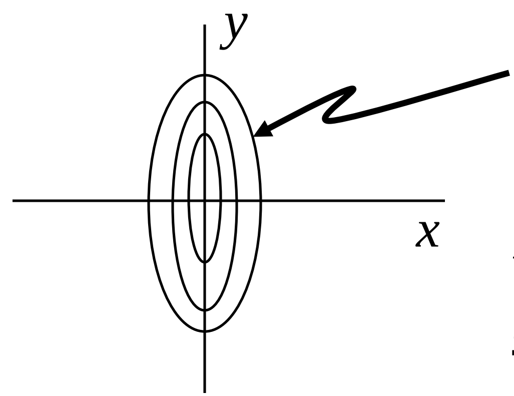
In other words: conditioning doesn't change the PDF!!!

$$p_{Y|X=x}(y \mid x) = \frac{p_{XY}(x, y)}{p_X(x)} = p_Y(y)$$

$$p_{X|Y=y}(x \mid y) = \frac{p_{XY}(x, y)}{p_Y(y)} = p_X(x)$$

Independent and Dependent Gaussian PDFs

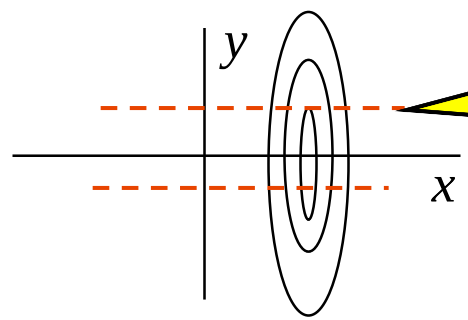
Independent
(zero mean)



Contours of $p_{XY}(x,y)$.

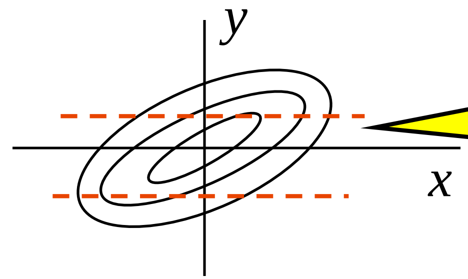
If X & Y are independent, then the contour ellipses are aligned with either the x or y axis

Independent
(non-zero mean)



Different slices give same normalized curves

Dependent



Different slices give different normalized curves

An "Independent RV" Result

RV's X & Y are independent if:

$$p_{XY}(x,y) = p_X(x)\,p_Y(y)$$

Here's why:

$$p_{Y|X=x}(y\,|\,x) = \frac{p_{XY}(x,y)}{p_X(x)} = \frac{\cancel{p_X(x)}\,p_Y(y)}{\cancel{p_X(x)}} = p_Y(y)$$

Characterizing RVs

- PDF tells everything about an RV
  - but sometimes they are “more than we need/know”
- So… we make due with a few Characteristics
  - Mean of an RV (Describes the centroid of PDF)
  - Variance of an RV (Describes the spread of PDF)
  - Correlation of RVs (Describes “tilt” of joint PDF)

Mean = Average = Expected Value

Symbolically: $E\{X\}$

Motivating Idea of Mean of RV

Motivation First w/ "Data Analysis View"

Consider RV X = Score on a test   Data: $x_1, x_2, \ldots x_N$

Possible values of RV X : $V_0 \quad V_1 \quad V_2 \ldots \quad V_{100}$

$0 \quad 1 \quad 2 \quad \ldots \; 100$

$$\text{Test Average} = \bar{x} = \frac{\sum_{i=1}^{N} x_i}{N} = \frac{N_0 V_0 + N_1 V_1 + \ldots N_n V_{100}}{N} = \sum_{i=0}^{100} V_i \frac{N_i}{N}$$

$\frac{N_i}{N} \approx P(X = V_i)$

N_i = # of scores of value V_i

$N = \sum_{i=1}^{n} N_i$ (Total # of scores)

This is called Data Analysis View — Statistics

But it motivates the Data Modeling View — Probability

Theoretical View of Mean

Data Analysis View leads to Probability Theory:

Data Modeling

- For Discrete random Variables :

$$E\{X\} = \sum_{n=1}^{n} x_i P_X(x_i)$$

Probability Function

- This Motivates form for Continuous RV:

$$E\{X\} = \int_{-\infty}^{\infty} x \, p_X(x) dx$$

Probability Density Function

Notation: $E\{X\} = \overline{X}$

Shorthand Notation

Aside: Probability vs. Statistics

Probability Theory

» Given a PDF Model

» Describe how the data will likely behave

$$E\{X\} = \int_{-\infty}^{\infty} x \, p_X(x) dx$$

Dummy Variable

PDF

There is no DATA here!!!

The PDF models how the data will likely behave

Statistics

» Given a set of Data

» Determine how the data did behave

$$Avg = \frac{1}{N} \sum_{i=1}^{N} x_i$$

Data

There is no PDF here!!!

The Statistic measures how the data did behave

$\approx$

"Law of Large Numbers"

Variance of RV

There are similar Data vs. Theory Views here…

But let's go right to the theory!!

Variance: Characterizes how much you expect the RV to Deviate Around the Mean

Variance: $\sigma^2 = E\{(X - m_X)^2\}$

$$= \int (x - m_X)^2 p_X(x) dx$$

Note : If zero mean…

$$\sigma^2 = E\{X^2\}$$

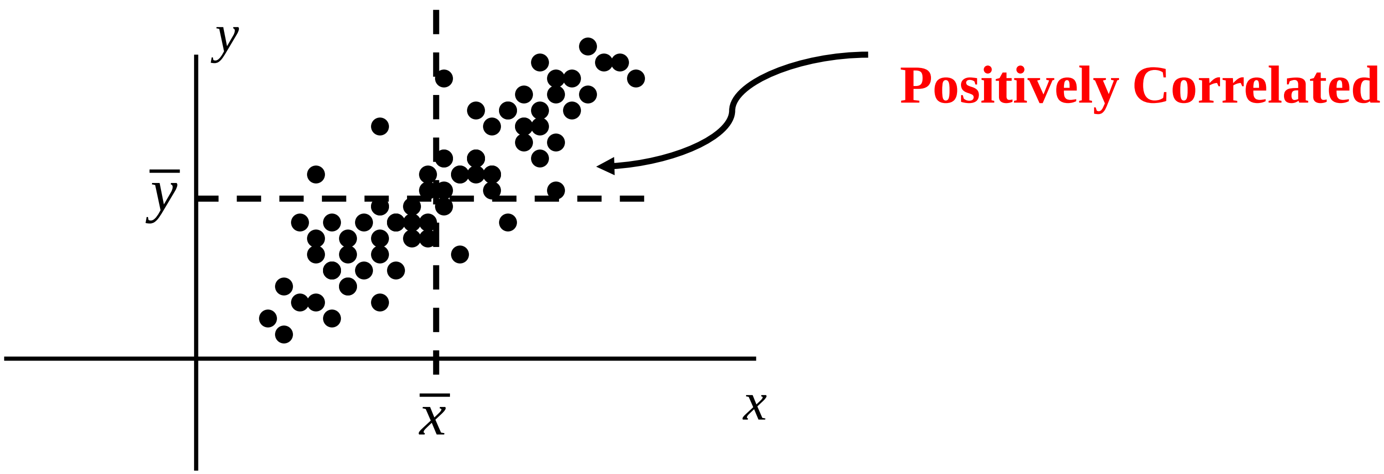
$$= \int x^2 p_X(x) dx$$

Motivating Idea of Correlation

Motivate First w/ Data Analysis View

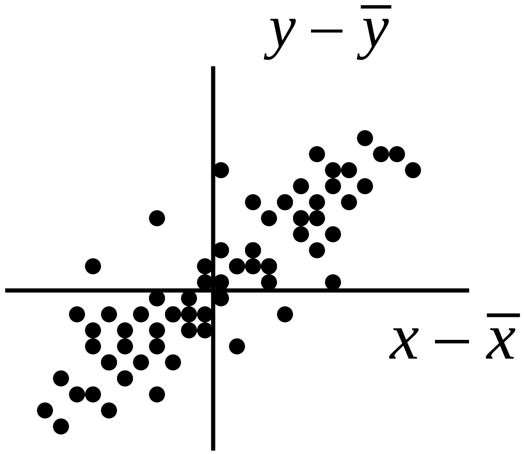
Consider a random experiment that observes the outcomes of two RVs:

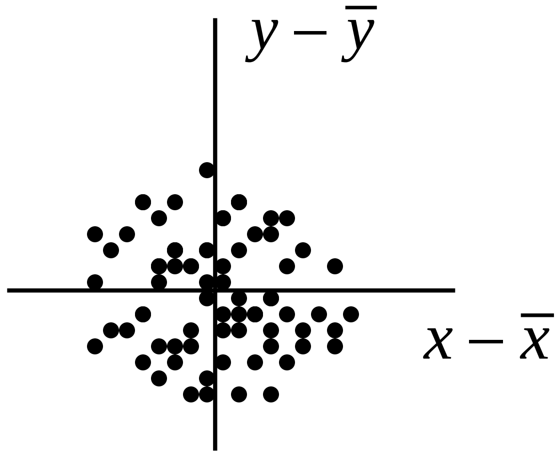
Example: 2 RVs *X* and *Y* representing height and weight, respectively

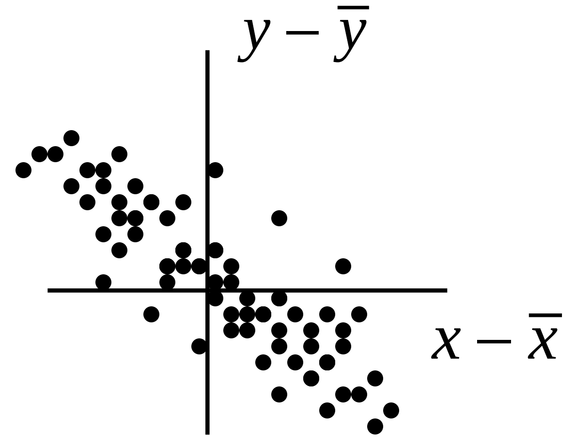


Illustrating 3 Main Types of Correlation

Data Analysis View: $C_{xy} = \frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})(y_i - \bar{y})$







Positive Correlation
"Best Friends"

Zero Correlation
i.e. uncorrelated
"Complete Strangers"

Negative Correlation
"Worst Enemies"

GPA
&
Starting Salary

Height
&
$ in Pocket

Student Loans
&
Parents' Salary

Prob. Theory View of Correlation

To capture this, define Covariance :

$$\sigma_{XY} = E\{(X - \overline{X})(Y - \overline{Y})\}$$

$$\sigma_{XY} = \int\int (x - \overline{X})(y - \overline{Y})\, p_{XY}(x, y)\, dx\, dy$$

If the RVs are both Zero-mean : $\sigma_{XY} = \mathrm{E}\{XY\}$

If $X = Y$: $\sigma_{XY} = \sigma_X^2 = \sigma_Y^2$

If X & Y are independent, then: $\sigma_{XY} = 0$

If $\sigma_{XY} = E\{(X - \overline{X})(Y - \overline{Y})\} = 0$

Then… Say that *X* and *Y* are **<u>"uncorrelated"</u>**

If $\sigma_{XY} = E\{(X - \overline{X})(Y - \overline{Y})\} = 0$

Then $\underbrace{E\{XY\}} = \overline{X}\,\overline{Y}$

Called <u>"Correlation of *X* & *Y*"</u>

So… RVs *X* and *Y* are said to be uncorrelated

if $\sigma_{XY} = 0$

or equivalently… if $E\{XY\} = E\{X\}E\{Y\}$

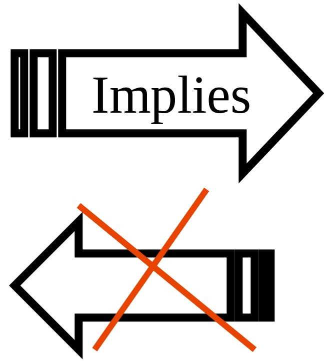
Independence vs. Uncorrelated

X & Y are Independent

$f_{XY}(x, y)$

$= f_X(x) f_Y(y)$

PDFs Separate

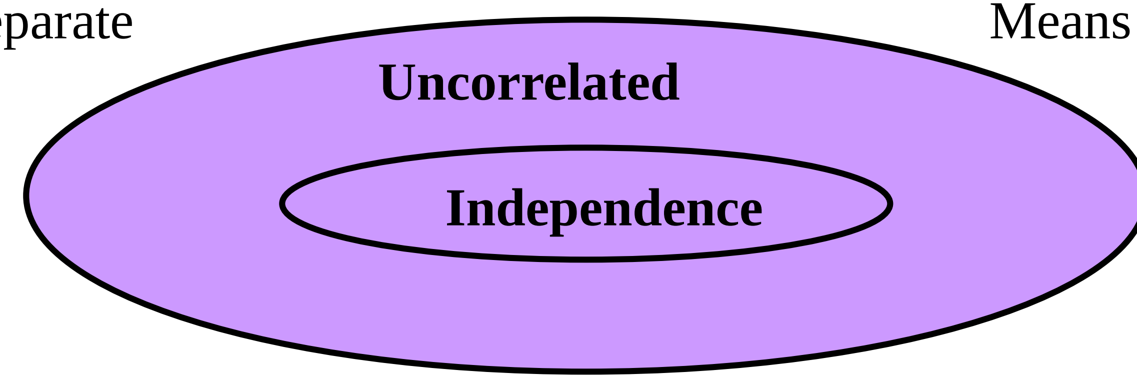


X & Y are Uncorrelated

$E\{XY\}$

$= E\{X\}E\{Y\}$

Means Separate



INDEPENDENCE IS A STRONGER CONDITION !!!!

Confusing Covariance and Correlation Terminology

Covariance : $$\sigma_{XY} = E\{(X - \bar{X})(Y - \bar{Y})\}$$

Correlation : $$E\{XY\}$$

Same if zero mean

Correlation Coefficient : $$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

$$-1 \le \rho_{XY} \le 1$$

Covariance and Correlation For Random Vectors…

$$\mathbf{x} = [\, X_1 \; X_1 \; \cdots \; X_N \,]^T$$

Correlation Matrix :

$$\mathbf{R_x} = E\{\mathbf{x}\mathbf{x}^T\} = \begin{bmatrix} E\{X_1 X_1\} & E\{X_1 X_2\} & \cdots & E\{X_1 X_N\} \\ E\{X_2 X_1\} & E\{X_2 X_2\} & \cdots & E\{X_2 X_N\} \\ \vdots & \vdots & \ddots & \vdots \\ E\{X_N X_1\} & E\{X_N X_2\} & \cdots & E\{X_N X_N\} \end{bmatrix}$$

Covariance Matrix :

$$\mathbf{C_x} = E\{(\mathbf{x} - \overline{\mathbf{x}})(\mathbf{x} - \overline{\mathbf{x}})^T\}$$

A Few Properties of Expected Value

$$E\{X+Y\}=E\{X\}+E\{Y\}$$

$$E\{aX\}=aE\{X\}$$

$$E\{f(X)\}=\int f(x)p_X(x)dx$$

$$\text{var}\{X+Y\}=\begin{cases}\sigma_X^2+\sigma_Y^2+2\sigma_{XY} \\ \sigma_X^2+\sigma_Y^2, \quad \text{if } X \,\&\, Y \text{ are uncorrelated}\end{cases}$$

$$\text{var}\{aX\}=a^2\sigma_X^2$$

$$\text{var}\{a+X\}=\sigma_X^2$$

$$\text{var}\{X+Y\}=E\left\{\left(X+Y-\overline{X}-\overline{Y}\right)^2\right\}$$

$$=E\left\{\left(X_z+Y_z\right)^2\right\} \quad \text{where } X_z=X-\overline{X}$$

$$=E\left\{\left(X_z\right)^2+\left(Y_z\right)^2+2X_zY_z\right\}$$

$$=E\left\{\left(X_z\right)^2\right\}+E\left\{\left(Y_z\right)^2\right\}+2E\left\{X_zY_z\right\}$$

$$=\sigma_X^2+\sigma_Y^2+2\sigma_{XY}$$

Joint PDF for Gaussian

Let $\mathbf{x} = [X_1\ X_2\ \ldots\ X_N]^T$ be a vector of random variables. These random variables are said to be jointly Gaussian if they have the following PDF

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{N}{2}}\sqrt{\det(\mathbf{C}_x)}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu}_x)^T \mathbf{C}_x^{-1}(\mathbf{x}-\boldsymbol{\mu}_x)\right\}$$

where $\boldsymbol{\mu}_x$ is the mean vector and $\mathbf{C}_x$ is the covariance matrix:

$$\boldsymbol{\mu}_x = E\{\mathbf{x}\} \qquad \mathbf{C}_x = E\{(\mathbf{x}-\boldsymbol{\mu}_x)(\mathbf{x}-\boldsymbol{\mu}_x)^T\}$$

For the case of two jointly Gaussian RVs X_1 and X_2 with

$E\{X_i\} = \mu_i$ $\quad$ $\text{var}\{X_i\} = \sigma_i^2$ $\quad$ $E\{(X_1 - \mu_1)(X_2 - \mu_2)\} = \sigma_{12}$ $\quad$ $\rho = \sigma_{12}/(\sigma_1 \sigma_2)$

Then…

$$p(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right]\right\}$$

It is easy to verify that X_1 and X_2 are uncorrelated (and independent!) if $\rho = 0$

Linear Transform of Jointly Gaussian RVs

Let $\mathbf{x} = [X_1\ X_2\ \ldots\ X_N]^T$ be a vector of jointly Gaussian random variables with mean vector $\boldsymbol{\mu}_x$ and covariance matrix $\mathbf{C}_x$…

Then the linear transform $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{b}$ is also jointly Gaussian with

$$\boldsymbol{\mu}_y = E\{\mathbf{y}\} = \mathbf{A}\boldsymbol{\mu}_x + \mathbf{b}$$

$$\mathbf{C}_y = E\{(\mathbf{y} - \boldsymbol{\mu}_y)(\mathbf{y} - \boldsymbol{\mu}_y)^T\} = \mathbf{A}\mathbf{C}_x\mathbf{A}^T$$

A special case of this is the **sum of jointly Gaussian RVs**… which can be handled using $\mathbf{A} = [1\ \ 1\ \ 1\ \ \ldots\ \ 1]$

Moments of Gaussian RVs

Let X be zero mean Gaussian with variance σ^2

Then the moments $E\{X^k\}$ are as follows:

$$E\{X^k\} = \begin{cases} 1\cdot 3\cdots(k-1)\sigma^k, & k \text{ even} \\ 0, & k \text{ odd} \end{cases}$$

Let X_1 X_2 X_3 X_4 be any four jointly Gaussian random variables with zero mean

Then…

$$E\{X_1X_2X_3X_4\} = E\{X_1X_2\}\mathrm{E}\{X_3X_4\} + E\{X_1X_3\}\mathrm{E}\{X_2X_4\} + E\{X_1X_4\}\mathrm{E}\{X_2X_3\}$$

Note that this can be applied to find $E\{X^2Y^2\}$ if X and Y are jointly Gaussian

Chi-Squared Distribution

Let $X_1\ X_2\ \ldots\ X_N$ be a set of zero-mean independent jointly Gaussian random variables each with unit variance.

Then the RV $Y = X_1^2 + X_2^2 + \ldots + X_N^2$ is called a chi-squared (χ^2) RV of N degrees of freedom and has PDF given by

$$p(y) = \begin{cases} \dfrac{1}{2^{N/2}\Gamma(N/2)} y^{(N/2)-1} e^{-y/2}, & y \geq 0 \\ 0, & y < 0 \end{cases}$$

For this RV we have that:

$$E\{Y\} = N \quad \text{and} \quad \text{var}\{Y\} = 2N$$