

Problem Statements

1. Imagine that a remote computer sends a 1 or a 0 by sending either $p(t)$ or $-p(t)$, respectively... where the pulse $p(t)$ is given by:

$$p(t) = \begin{cases} 1, & 0 \leq t \leq T \\ 0, & \text{otherwise} \end{cases}$$

When the pulse is received it is corrupted by Gaussian noise that is added to it. Assume that this noise has zero mean and has variance of 1. Suppose that you wish to measure a single value within a received pulse and use that to determine if a 1 or a 0 was sent. Then the value that you measure would be a Gaussian random variable Z modeled as follows:

$$Z = \pm 1 + V$$

where V is a Gaussian random variable with zero mean and variance of 1. Say that you will decide that a 1 was sent if the measured Z is such that $Z > 0$ and otherwise will decide that a 0 was sent.

Consider the case where a 1 was sent (thus $Z = +1 + V$) and find the probability of making an error.

2. For the scenario in Problem #1, we now consider talking 10 samples (Z_1, Z_2, \dots, Z_{10}) within a single pulse and computing the data average of the measured Z values:

$$M = \frac{1}{10} \sum_{i=1}^{10} Z_i$$

Now you would decide that a 1 was sent if M is such that $M > 0$ and otherwise will decide that a 0 was sent. Consider the case where a 1 was sent (thus $Z_i = +1 + V_i$ and assume that the noise RVs V_i are uncorrelated) and find the probability of making an error.

3. The following is a contrived problem that is intended to show how probability theory is used in engineering (Since it is contrived, I honestly don't know if this model realistically describes the manufacture of resistors – I'd think it isn't!). Imagine that you are analyzing a machine that makes 1 k Ω resistors and that a model for the resistance value of each resistor produced is given by

$$R_{actual} = \alpha(T - T_{nominal}) + 1000$$

where T is the actual temperature (in $^{\circ}\text{C}$) of the material used in making the resistors, $T_{nominal}$ is the nominal expected temperature (in $^{\circ}\text{C}$) of the material (imagine that a control system is in place that attempts to maintain the temperature at $T_{nominal}$), and α is a known parameter that characterizes the sensitivity of the actual resistance value to changes in the temperature. Suppose we are told by the temperature control system engineers that their analysis shows that a good model for the temperature T is that it is a zero-mean Gaussian RV with some specific standard deviation σ_T . Suppose that the sensitivity parameter is known to be $\alpha = 10 \text{ } \Omega/^{\circ}\text{C}$. What specification should you give to the control system engineers for the standard deviation σ_T in order to ensure that 95% of the manufactured resistors are within 1% of the nominal 1k Ω resistance?

Solutions

1. This problem and the next one show the value of probability theory to the design of data transferal systems – without probability theory an engineer can not properly make design decisions for such problems.

In this problem you are to determine the probability of error that occurs when a 1 is sent. Thus, you know that the measured RV Z is given by $Z = +1 + V$ where V is a zero-mean Gaussian RV with variance of 1. If there were no noise in the system there would be no error because Z would be 1; with noise in the system the value that Z takes on can (in theory) be any real number (although some of those numbers are very unlikely). When the value of Z is below 0 we make an error, which occurs when $V \leq -1$; so we need to determine the probability that $V \leq -1$. We have a probability model for V that says V is Gaussian with zero-mean and variance of 1, so the PDF of V is given by

$$p_V(v) = \frac{1}{\sqrt{2\pi}} e^{-v^2/2}$$

Now, to find the probability of an error we find probability that $V \leq -1$ as follows:

$$\Pr(\text{error}) = \Pr(V \leq -1)$$

$$= \int_{-\infty}^{-1} p_V(v) dv$$

$$= \int_{-\infty}^{-1} \frac{1}{\sqrt{2\pi}} e^{-v^2/2} dv$$

$$= \int_1^{\infty} \frac{1}{\sqrt{2\pi}} e^{-v^2/2} dv$$

$$= 1 - \int_{-\infty}^1 \frac{1}{\sqrt{2\pi}} e^{-v^2/2} dv$$

By definition of how probability is computed from a PDF

Using specific form for the PDF of V – This is the shaded area shown on the left side in the top part of Figure 1

This is the shaded area shown on the right side in the top part of Figure 1. The two shaded areas in the top part of Figure 1 are equal due to the symmetry of the PDF

where some of the manipulations are shown in Figure 1...

From the bottom part of the Figure 1 we see that we can get what we want (the right hand shaded part) by subtracting the left shaded part from 1 (the total area = 1).

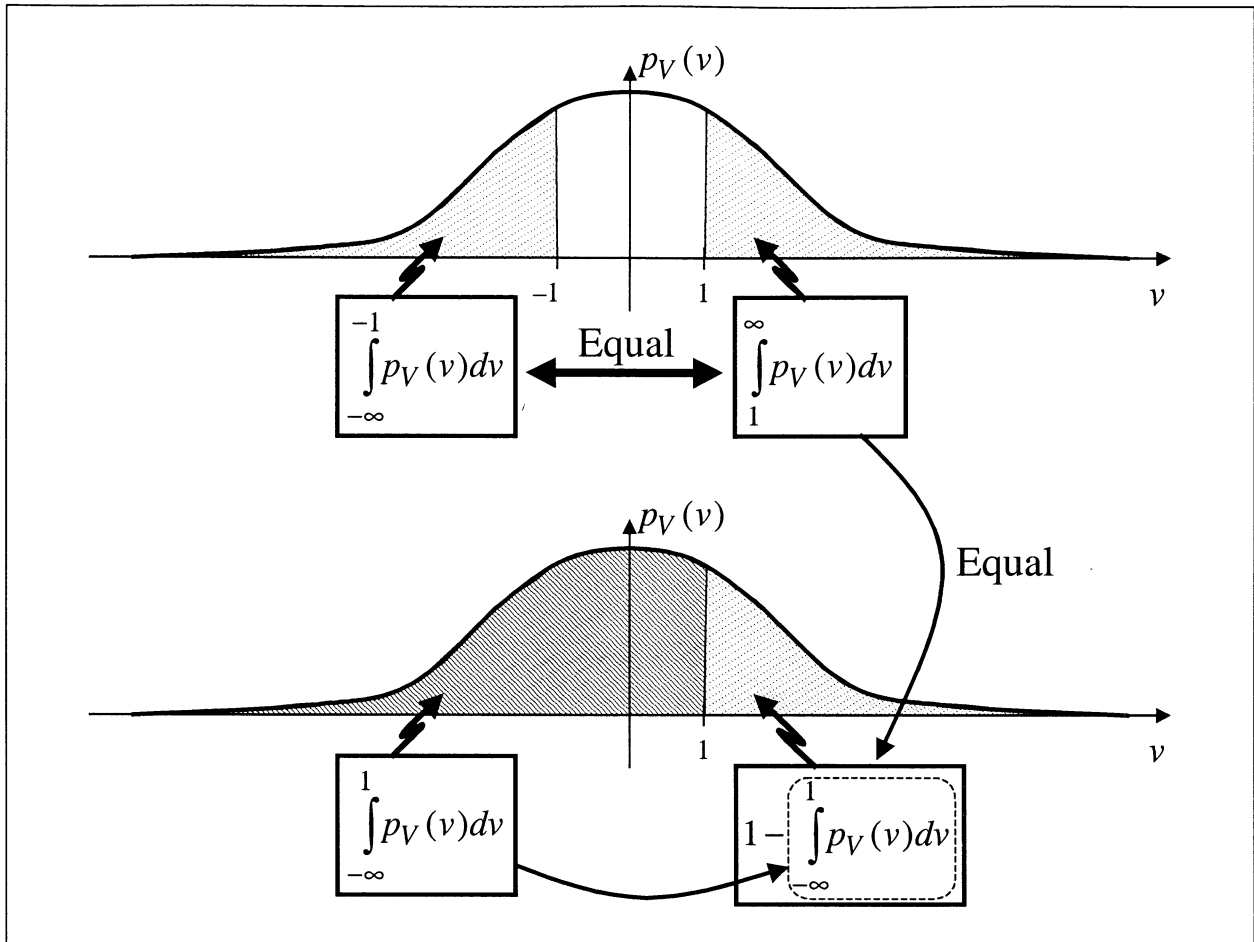


Figure 1: Relationships between various areas under the PDF of V

The value of the integral in this last line is something that is given in a table in EVERY introductory textbook on probability, one of which is shown in Figure 2. Looking at the line in the table for 1.0 gives a value in the table of 0.8413, which after subtraction from 1 gives

$$\Pr(\text{error}) = 1 - 0.8413 = 0.1587$$

This says that if we use this simple method of deciding if a 1 or 0 was sent that we would expect to make an error roughly on 16% of the bits that are sent... that is terrible performance. Without this “predictive” analysis using probability theory how would you know that this scheme works so poorly???!!!!!

NORMAL DISTRIBUTION FUNCTION

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-\frac{1}{2}t^2} dt$$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Figure 2: Table giving values for area under Gaussian PDF

#2. The scheme to be analyzed here is the following. Instead of taking a single sample of the signal-plus-noise you take 10 samples and do a “data average” to create the value M , which is then used to decide: if $M > 0$ you decide that a 1 was sent. Once you have the PDF of M the analysis of this problem is pretty much the same as for #4:

$$\begin{aligned}\Pr(\text{error}) &= \Pr(M \leq 0) \\ &= \int_{-\infty}^0 p_M(m) dm\end{aligned}$$

When a 1 has been sent, each sample is given by $Z_i = 1 + V_i$ so we can write M in terms of the random variables V_i as follows:

$$\begin{aligned}M &= \frac{1}{10} \sum_{i=1}^{10} Z_i \\ &= \frac{1}{10} \sum_{i=1}^{10} (1 + V_i) \\ &= \frac{1}{10} \sum_{i=1}^{10} 1 + \frac{1}{10} \sum_{i=1}^{10} V_i \\ &= 1 + \underbrace{\frac{1}{10} \sum_{i=1}^{10} V_i}_{=\tilde{V}}\end{aligned}$$

Thus, we can analyze the problem exactly as before except now we need to ask what is the probability that the RV \tilde{V} is less than -1 . But what is the PDF of \tilde{V} ? From probability theory we know that the sum of Gaussian RV's is also a Gaussian RV. Because a Gaussian PDF is completely defined by its mean and variance, we need to determine the mean and variance of \tilde{V} .

To find the mean we use properties of the expected value:

$$E\{\tilde{V}\} = E\left\{\frac{1}{10} \sum_{i=1}^{10} V_i\right\} = \frac{1}{10} \sum_{i=1}^{10} E\{V_i\} = 0$$

where the final zero comes from the fact that each noise RV V_i is zero mean.

The variance can be found as follows:

$$\begin{aligned}
\text{Var}\{\tilde{V}\} &= E\{\tilde{V}^2\} = E\left\{\left[\frac{1}{10}\sum_{i=1}^{10}V_i\right]^2\right\} = \frac{1}{100}E\left\{\left[\sum_{i=1}^{10}V_i\right]^2\right\} \\
&= \frac{1}{100}E\left\{\underbrace{V_1V_1 + V_2V_2 + \dots + V_{10}V_{10}}_{\text{"auto-terms"}} + \underbrace{V_1V_2 + V_1V_3 + \dots + V_{10}V_9}_{\text{"cross-terms"}}\right\} \\
&= \frac{1}{100}\left[\underbrace{E\{V_1^2\} + E\{V_2^2\} + \dots + E\{V_{10}^2\}}_{\text{each is 1 by problem definition}} + \underbrace{E\{V_1V_2\} + E\{V_1V_3\} + \dots + E\{V_{10}V_9\}}_{\text{each is zero by problem statement that noise RVs are uncorrelated and zero mean}} \right] \\
&= \frac{1}{10}
\end{aligned}$$

Thus we see that the \tilde{V} RV is zero-mean Gaussian with variance of 1/10 (thus it varies less than the V random variable).

So now we can find the desired probability:

$$\begin{aligned}
\text{Pr}(\text{error}) &= \text{Pr}(\tilde{V} \leq -1) \\
&= 1 - \int_{-\infty}^{-1} p_{\tilde{V}}(v) dv && \text{By definition of how probability is computed from a PDF} \\
&= 1 - \int_{-\infty}^{-1} \frac{1}{\sqrt{2\pi} \frac{1}{\sqrt{10}}} e^{-v^2/(2/10)} dv && \text{Plug in the specific PDF} \\
&= 1 - \sqrt{10} \int_{-\infty}^{-1} \frac{1}{\sqrt{2\pi}} e^{-(v\sqrt{10})^2/2} dv && \text{Simplify using algebra} \\
&= 1 - \int_{-\infty}^{-\sqrt{10}} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy && \text{Change variable in the integral: let } y = \text{sqrt}(10)v \text{ so that } dv = dy/\text{sqrt}(10) \text{ and the upper limit changes to } \text{sqrt}(10) \\
&= 1 - 0.9992 = 8 \times 10^{-4} && \text{This is in the form that is tabulated... so look up for } \text{sqrt}(10) = 3.16
\end{aligned}$$

The answer is:

$$\text{Pr}(\text{error}) = 1 - 0.9992 = 8 \times 10^{-4}$$

which is a much better performance... still not as good as what we would want, though. But, this theory shows us the road to better performance: use a data average with more than 10 values. As you increase the number of samples you average, the probability of error will decrease. So if you have a desired probability of error that you want to achieve, you can use probability theory to determine how many samples you should average together!!!! ... and that is engineering!!!! In Figure 3 I show a plot that illustrates this (notice the logarithmic scale on the probability axis – that is common in this application because the distinction between, say, 10^{-3} and 10^{-6} is important and just wouldn't show up on a linear axis. Note that to get a prob. of error of 10^{-6} we would need to average about 18 samples together.

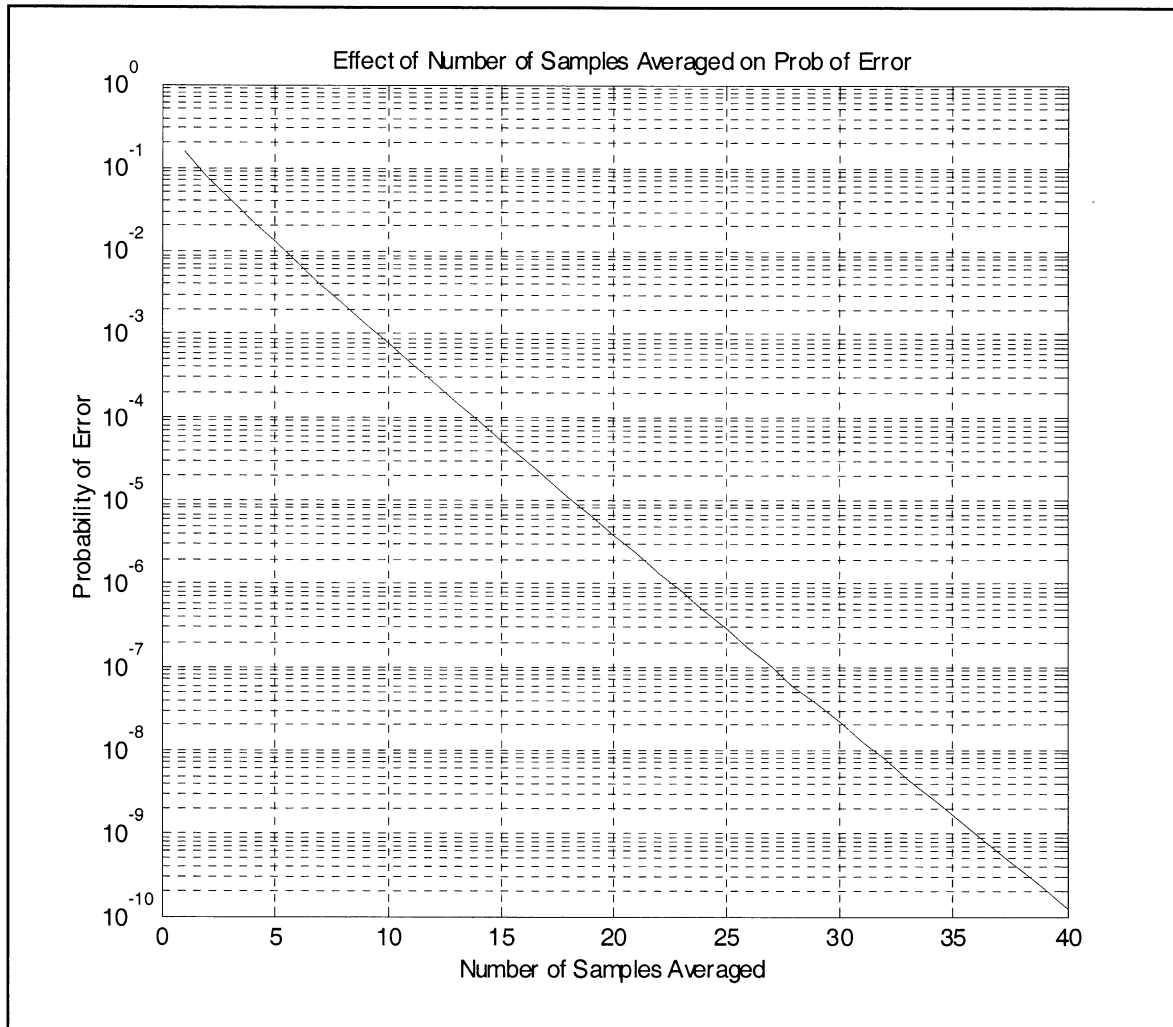


Figure 3: Effect of number of samples averaged on the probability of an error.

#3. The control system engineers have told you that they have modeled the temperature T as Gaussian with mean T_{nominal} and some unspecified standard deviation Presumably they can design their control system to achieve a range of standard deviations – the smaller the std. dev., the better their control system works because a small std. dev. indicates small fluctuations around the temperature point they are trying to maintain. Our job (as the engineers in charge of making

the resistors) is to tell them how much (or how little) fluctuation we can stand and still manufacture a resistor to within a 10% tolerance (defined here as “95% of manufactured resistors have no worse than a 10% error”).

For ease of analysis: define $\Delta T = T - T_{\text{nominal}}$. Then, because T is a Gaussian RV and T_{nominal} is a number, ΔT is a Gaussian RV with the following mean:

$$E\{\Delta T\} = E\{T - T_{\text{nominal}}\} = E\{T\} - T_{\text{nominal}} = T_{\text{nominal}} - T_{\text{nominal}} = 0$$

where we have used properties of expected value and the fact that T has mean of T_{nominal} .

Thus, we know that ΔT is a Gaussian RV with zero mean. What is its variance? That is found using

$$\text{Var}\{\Delta T\} = E\{\Delta T^2\} = \underbrace{E\{(T - T_{\text{nominal}})^2\}}_{=\text{Var}\{T\} \text{ by definition}}$$

Thus, we see that ΔT has the same variance as T (a general result says that adding a number to an RV changes its mean but not its variance).

Now, because ΔT is an RV, the resistance value R_{actual} (which depends on ΔT) is also an RV. A general result says that if X is a Gaussian RV, then $aX+b$ is also Gaussian. What is its mean and variance? Find this as follows:

$$E\{R_{\text{actual}}\} = E\{\alpha\Delta T + 1000\} = \alpha \underbrace{E\{\Delta T\}}_{=0} + 1000 = 1000$$

and

$$\text{Var}\{R_{\text{actual}}\} = E\{(R_{\text{actual}} - 1000)^2\} = E\{(\alpha\Delta T)^2\} = \alpha^2 \underbrace{E\{(\Delta T)^2\}}_{=\text{Var}(T)=\sigma_T^2}$$

Thus, we know that the actual resistance value is a Gaussian RV with mean of 1000 ohms and a standard deviation of σ_T (yet-to-be-specified). Thus, we have a PDF model for the actual resistance that looks like what is shown in Figure 4; note that if σ_T is too big then the probability of the resistance being inside the 1% tolerance is less than 0.95 (that is, less than 95% of the resistors will be within spec).

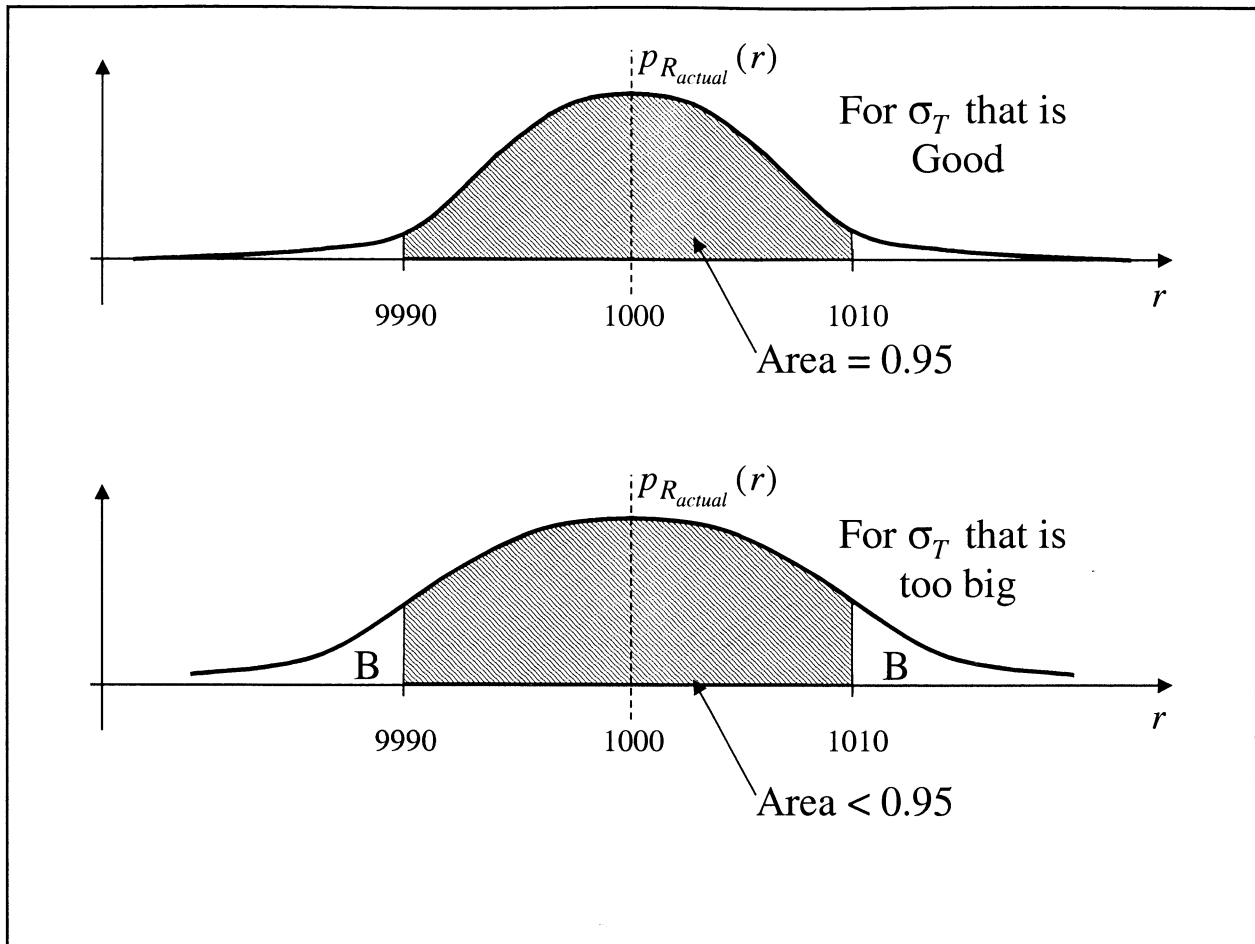


Figure 4: PDF of resistance for two cases of the temperature standard deviation

Now the question becomes, what value of σ_T is needed to ensure that we meet the requirement that 95% of the resistors are within the 1% tolerance? Let B be the area in each of the tails shown in Figure 4; then we need that $1 - 2B = 0.95$ or $B = 0.025$. So we need to find the value of the σ_T that causes $P(R_{\text{actual}} > 1010) = 0.025$... But this is equivalent to first centering the PDF around 0 rather than around 1000: then we need to find the value of the σ_T that causes $P(R_{\text{actual}} - 1000 > 10) = 0.025$ or

$$\begin{aligned}
 0.025 &= 1 - \int_{-\infty}^{10} \frac{1}{\sqrt{2\pi}\sigma_T} e^{-r^2/(2\sigma_T^2)} dr \\
 &= 1 - \int_{-\infty}^{10/\sigma_T} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy
 \end{aligned}$$

or

$$0.975 = \int_{-\infty}^{10/\sigma_T} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$$

From the table in Figure 2 we see that we need $10/\sigma_T = 1.96$ or 5.1°C which is probably not too terribly difficult for a control system!!!