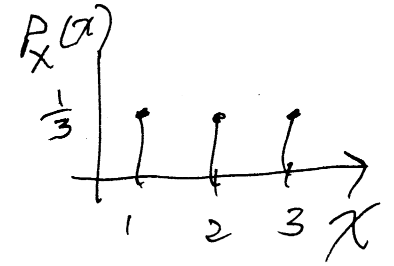
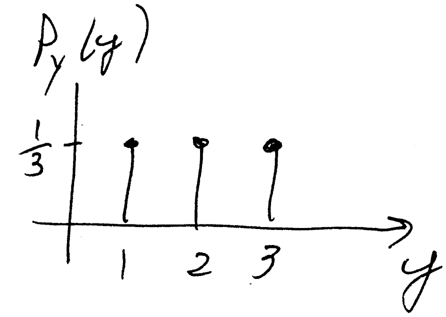
A

Ex. #1 Partially Observed Dice

X: 3 sided "Die" 1, 2, 3 ~~Pro~~ Prob. of each $= \frac{1}{3}$



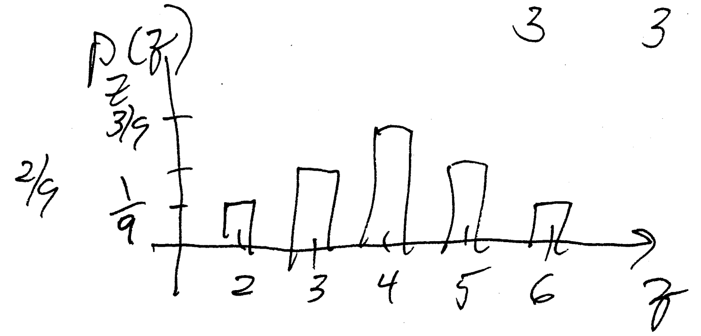
Y: another one of these



$Z = X + Y$

1	1
2	2
3	3

9 outcomes
prob. $= 1/9$



$\hat{z}$

Lets say we want guess what value z will take.

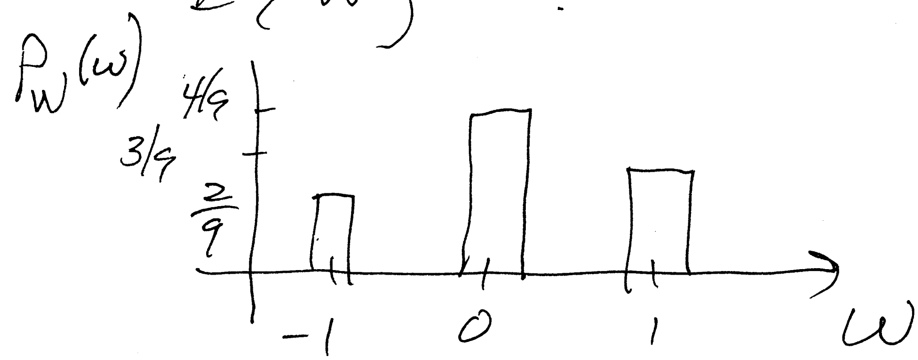
Win \$1 if correct
" \$0 if $|\hat{z} - z| = 1$
" $-\$1$ if $|\hat{z} - z| \geq 2$

$\uparrow W$

Strategy #1 Choose $\hat{z} = E\{z\}$
$= 4$

Expected Win ?

$E\{W\} = ?$



$$E\{W\} = \sum_{w=-1}^{1} w P_W(w)$$

$$= -1 \times \tfrac{2}{9} + 0 \times 4/9 + 1 \times 3/9 = 1/9$$

Strategy #2 suppose somehow you know the value that Y takes on!

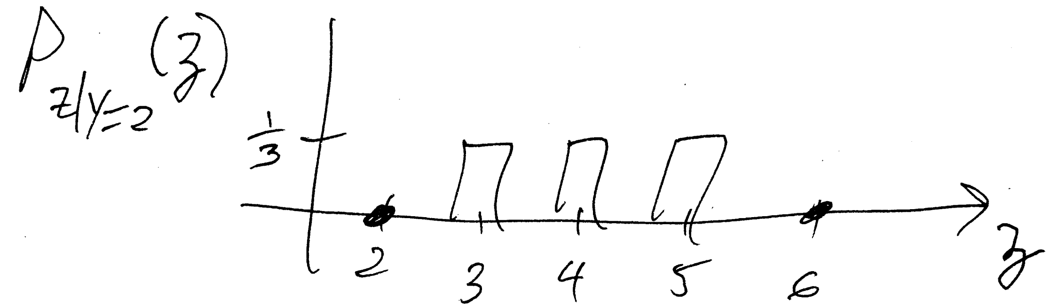
$Z = X + Y$

How to guess $\hat{z}$ now?

Know that $Y = y$

$$\hat{z}_{Y=y} = E\{Z \mid Y=y\}$$

c



etc.

$$\hat{z}_{Y=y} = E\{Z | Y = y\} = \begin{cases} 3 & y=1 \\ 4 & y=2 \\ 5 & y=3 \end{cases}$$

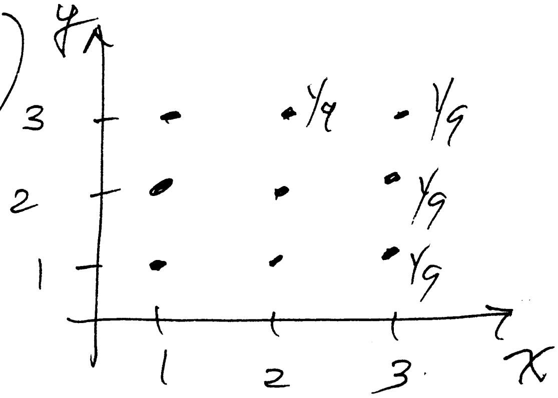
For this case what is $E\{W\}$

$$E\{W | Y = y\} = \begin{cases} \frac{1}{3}\times 1 + \frac{2}{3}\times 0 = \frac{1}{3}, & y=1 \\ \frac{1}{3} & y=2 \\ \frac{1}{3} & y=3 \end{cases}$$

$$E\{W\} = \frac{1}{3}\times\frac{1}{3} + \frac{1}{3}\times\frac{1}{3} + \frac{1}{3}\times\frac{1}{3} = \frac{1}{3}$$

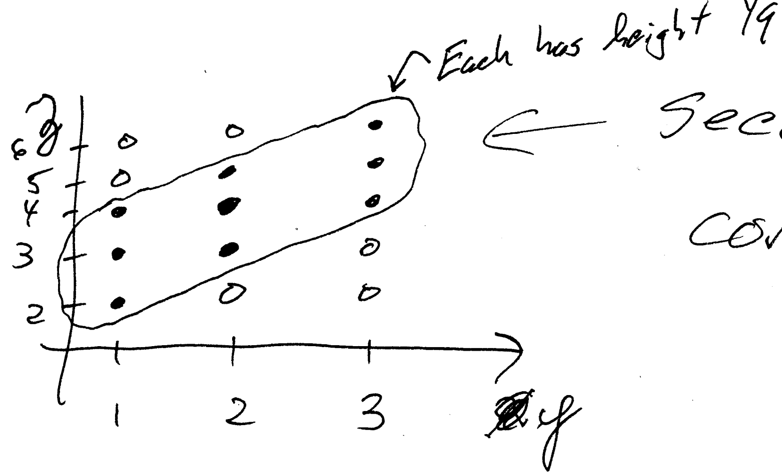
D

$P_{xy}(x,y)$



By observation

$P_{zy}(z,y)$



Seeing correlation

$$\sigma_{XY} = E\{(X-\bar{X})(Y-\bar{Y})\}$$

$$= E\{XY\} \underbrace{- \bar{X}E\{Y\} - \bar{Y}E\{X\} + \bar{X}\bar{Y}}_{-\bar{X}\bar{Y}}$$

$$E\{XY\} = \sum_{i=1}^{3} \sum_{j=1}^{3} x_i y_i P_{xy}(x_i, y_i)$$

$$= \frac{1}{9} \times 1 \times 1 + \frac{1}{9} \times 1 \times 2 + \dots$$

$$= \frac{36}{9} = 4$$

$$\bar{X}\bar{Y} = 4$$

$$\sigma_{XY} = 0$$

E

$$\sigma_{zy} = E\{(z-\bar{z})(\cancel{x}-\cancel{\bar{y}})\}$$

$$= E\{zx\} - \underset{\substack{\uparrow \\ 4}}{\bar{z}}\;\underset{\substack{\uparrow \\ 2}}{\cancel{\bar{y}}}$$

$$E\{z\cancel{y}\} = \sum_{y=1}^{3}\sum_{z=2}^{6} y\, z\, P_{zy}(z,y)$$

$$= \left(1\times2\times\tfrac{1}{9} + 1\times3\times\tfrac{1}{9} + 1\times4\times\tfrac{1}{9}\right) + \left(2\times3\times\tfrac{1}{9} + 2\times4\times\tfrac{1}{9} + 2\times5\times\tfrac{1}{9}\right.$$

$$\left. + \left(3\times4\times\tfrac{1}{9} + 3\times5\times\tfrac{1}{9} + 3\times6\times\tfrac{1}{9}\right)\right)$$

$$= 8\tfrac{2}{3} = \frac{26}{3}$$ ~~Non-Zero ⇒~~

$$\Rightarrow \sigma_{zy} = \frac{26}{3} - 4\times2 = \frac{2}{3} \text{ Non-Zero} \Rightarrow \text{Correlated}$$

Find Correlation Coefficient

$$\sigma_z^2 = E\{(z-\bar{z})^2\} = E\{(z-4)^2\}$$

$$= \sum_{z=2}^{6}(z-4)^2 P_z(z)$$

$$= 4/3 \qquad \Rightarrow \sigma_z = \frac{2}{\sqrt{3}}$$

P

$$\sigma_Y^2 = E\{(Y-\bar{Y})^2\} = E\{(Y-2)^2\}$$

$$= \sum_{y=1}^{3} (y-2)^2 \frac{1}{3}$$

$$= \frac{2}{3} \quad \Rightarrow \quad \sigma_Y = \frac{\sqrt{2}}{\sqrt{3}}$$

So the correlation coefficient is

$$\rho = \frac{\sigma_{ZY}}{\sigma_Y \sigma_Z} = \frac{(2/3)}{\frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{2}{\sqrt{3}}} = \frac{2}{\sqrt{2}\,2} = \frac{1}{\sqrt{2}}$$

$$\rho = 0.701$$

↑ fairly close to 1

⇒ moderate correlation that can be exploited