Frequency Measurement in Noise

Porat Section 6.5

Frequency Meas. in Noise Problem

Want to now look at the effect of noise on using the DFT to measure the frequency of a sinusoid.

Assume Complex White Noise
Gaussian, Zero-Mean
Variance: $\sigma_v^2 = \gamma_v$

Consider single complex sinusoid case:

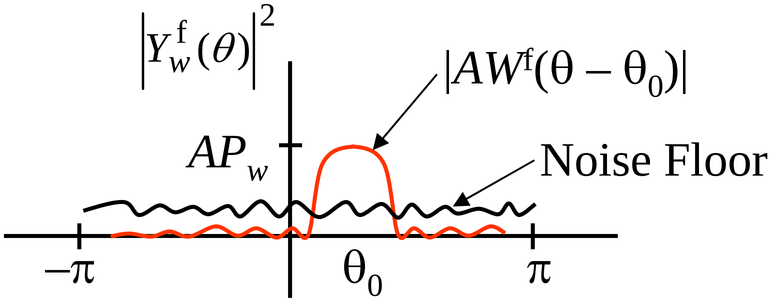
$$y[n] = Ae^{j\theta_0 n} + v[n], \quad 0 \le n \le N-1$$

Define: Input Signal-to-Noise Ratio (SNR):

$$SNR_i = \frac{\text{signal power}}{\text{noise power}} = \frac{A^2}{\sigma_v^2} \qquad \text{In dB}: \; 10\log_{10}\left(\frac{A^2}{\sigma_v^2}\right)$$

Model for Windowed DTFT of Received Signal:

$$Y_w^f(\theta) = AW^f(\theta - \theta_0) + V_w^f(\theta)$$



Impact of Noise

1. Makes it difficult to "see" the signal peak
   - Need signal peak well above the noise floor
   - If not…. Might not detect presence of signal
2. Noise perturbs the peak location
   - Degrades accuracy of the frequency estimate

So Processing Needs To….

- First, Detect the Signal
  - Look for peaks in the DFT
- Then, Estimate the Frequency (and amplitude/phase)
  - Same as before

Need to do analysis to determine the performance of these two† processing tasks. ➔ (Use DTFT in analysis rather than DFT)

† We'll only consider Detection Performance (see Porat's Book or EE522 for Estimation).

Signal Detection Analysis

Goal: Analyze relationships between peak level in DTFT due to signal and the noise floor height to answer:

Q: What parameters determine how high the signal's peak is above the noise floor?

DTFT of Windowed Noisy Signal:

$$Y_w^{\text{f}}(\theta) = DTFT\left\{ w[n]\left(Ae^{j\theta_0 n} + v[n] \right) \right\}$$

$$= \underbrace{A\sum_{n=0}^{N-1} w[n]e^{j(\theta_0-\theta)n}}_{\text{Signal Part}} + \underbrace{\sum_{n=0}^{N-1} w[n]v[n]e^{-j\theta n}}_{\text{Noise Part}}$$

Signal Detection Analysis (pt. 2)

Signal part peaks at $\theta = \theta_0$, so look there:

$$Y_w^{\mathrm{f}}(\theta_0) = A\underbrace{\sum_{n=0}^{N-1} w[n]}_{} + \sum_{n=0}^{N-1} w[n]v[n]e^{-j\theta_0 n}$$

Peak Height = A "Boosted" by $\Sigma w[n]$

For Rect. Window this "Boost" is: $\sum_{n=0}^{N-1} w_R[n] = N$

Q: What is the boost for other windows?

Compare $\Sigma w[n]$ for other windows to that for the Rect window:

$$CG = \frac{\sum_{n=0}^{N-1} w[n]}{\sum_{n=0}^{N-1} w_R[n]} = \frac{\sum_{n=0}^{N-1} w[n]}{N}$$

Define Coherent Gain of Window

"Boost Lost" due to using a Non-Rect Window

- Note: $CG \leq 1$ ("=" for Rect. Window)
- CG nearly independent of N

Signal Detection Analysis (pt. 3)

Re-write DTFT Peak Using CG:

$$Y_w^f(\theta_0) = A(N \times CG) + \sum_{n=0}^{N-1} w[n]v[n]e^{-j\theta_0 n}$$

Impact of Signal on Peak Height

Impact of "Processing Length"

↑Length ➔ ↑Peak

Impact of "Window Shape"

"More Rect" ➔ ↑Peak

➔ Output Peak = (Input Amplitude)×(N•CG)

However, the noise floor also increases…. So we need a way to measure "Improvement"…. "Output SNR"

Signal Detection Analysis (pt. 4)

$$\text{Output SNR} = SNR_o = \frac{\text{Power of DTFT's Signal Peak}}{\text{DTFT Noise Power at Peak}}$$

DTFT Power at Peak:

“FOIL” The Terms

$$\left|Y_w^{\mathrm{f}}(\theta_0)\right|^2 = Y_w^{\mathrm{f}}(\theta_0) \times \overline{Y}_w^{\mathrm{f}}(\theta_0)$$

$$= \left(NA \times CG\right)^2 + 2NA \times CG \times \mathrm{Re}\left\{\sum_{n=0}^{N-1} w[n]v[n]e^{-j\theta_0 n}\right\}$$

$$+ \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} w[n]w[m]v[n]\overline{v}[m]e^{-j\theta_0 (n-m)}$$

Signal Part

Noise Part

Signal Detection Analysis (pt. 5)

Now… need to look at the average output power:

Expected Value of 1st noise term is zero because $E\{v[n]\}=0$

$$E\left\{\left|Y_w^f(\theta_0)\right|^2\right\} = (NA \times CG)^2 + \underbrace{\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} w[n]w[m]\underbrace{E\{v[n]\bar{v}[m]\}}_{\sigma_v^2 \delta[n-m]} e^{-j\theta_0(n-m)}}_{\sigma_v^2 \sum_{n=0}^{N-1} w^2[n]}$$

Autocorr. of White Noise

Use Sifting Prop.

Signal Peak's Power: $(NA \times CG)^2$

Noise Power @ Peak: $\sigma_v^2 \sum_{n=0}^{N-1} w^2[n]$

Signal Detection Analysis (pt. 6)

Now… Can write expression for "Output" SNR:

$$SNR_o = \frac{(NA \times CG)^2}{\sigma_v^2 \sum_{n=0}^{N-1} w^2[n]} = \frac{A^2 (N \times CG)^2}{\sigma_v^2 \sum_{n=0}^{N-1} w^2[n]} = N \times SNR_i \underbrace{\left[\frac{N(CG)^2}{\sum_{n=0}^{N-1} w^2[n]} \right]}_{=PG}$$

SNR_i (marks $\frac{A^2}{\sigma_v^2}$)

Now… To "simplify" define "Processing Gain" PG:

$$PG = \frac{N(CG)^2}{\sum_{n=0}^{N-1} w^2[n]} = \frac{N\left(\frac{1}{N}\sum_{n=0}^{N-1} w[n]\right)^2}{\sum_{n=0}^{N-1} w^2[n]} \longrightarrow \boxed{PG = \frac{\left(\sum_{n=0}^{N-1} w[n]\right)^2}{N \sum_{n=0}^{N-1} w^2[n]}}$$

$$\boxed{SNR_o = N \times PG \times SNR_i}$$

Measures Effect of Signal Environment (SNR_i)

Measures Effect of Window Type (i.e., Shape) (PG)

Measures Effect of Processing Length (Don't Count Zero-Pads!!!) (N)

Signal Detection Analysis (pt. 7)

Comments

- Generally Need $SNR_o \geq 14$ dB to ensure reliable detection!

- PG ≤ 1 (with "=" for Rect Window)

- Coherent Gain (CG) vs. Processing Gain (PG)
  - CG relates Peak Level to Signal Amp: $\boldsymbol{\textit{Peak Level} = N \times CG \times A}$
  - PG relates Peak's SNR to Signal SNR: $\boldsymbol{SNR_o = N \times PG \times SNR_i}$

- CG and PG are usually Specified in dB
  - CG in dB: $\mathbf{10 \log_{10}(CG)^2}$
  - PG in dB: $\mathbf{10 \log_{10} PG}$

Squared!
Because CG is an Amplitude Gain

Not Squared!
Because PG is a Power Gain

Signal Detection Analysis (pt. 8)

Another View of Output *SNR*

Recall an earlier equation for output SNR:

$$SNR_o = \frac{(NA \times CG)^2}{\sigma_v^2 \sum_{n=0}^{N-1} w^2[n]}$$

Consider (for ease) the Rect Window (CG = 1 and $\Sigma w^2[n] = N$) so…

$$SNR_o = \frac{N^2 A^2}{N\sigma_v^2} = \frac{N^2 \times (\text{Input Signal Power})}{N \times (\text{Input Noise Power})}$$

Signal Power Boosted by N^2

Noise Power Boosted only by N

Since the Signal is Boosted More Than the Noise, we get a Boost in SNR:

$$SNR_o = N \times SNR_i \qquad (\text{recall: PG} = 1 \text{ for Rect})$$

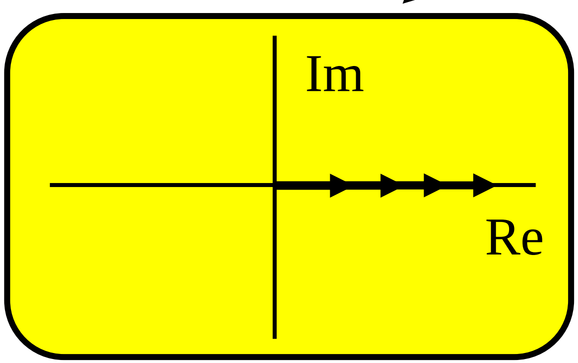
Signal Detection Analysis (pt. 9)

Yet Another View of Output *SNR*

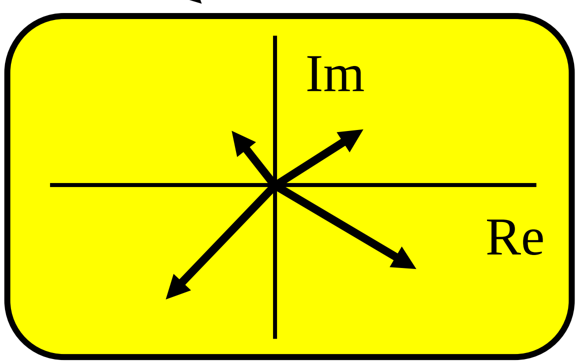
Recall this form for the DTFT at the peak:

$$Y_w^f(\theta)\Big|_{\theta=\theta_0} = \left[A \sum_{n=0}^{N-1} w[n] e^{j(\theta_0-\theta)n} \right]_{\theta=\theta_0} + \left[\sum_{n=0}^{N-1} w[n] v[n] e^{-j\theta n} \right]_{\theta=\theta_0}$$

$$= \underbrace{A \sum_{n=0}^{N-1} w[n] e^{j(0)n}} + \underbrace{\sum_{n=0}^{N-1} w[n] v[n] e^{-j\theta_0 n}}$$



Signal Terms Add "Coherently"
… Sum Grows Fast



Signal Terms Add "Incoherently"
… Sum Doesn't Grow *As* Fast

Signal Detection Analysis (pt. 9)

Impact of Actually Using DFT rather than DTFT

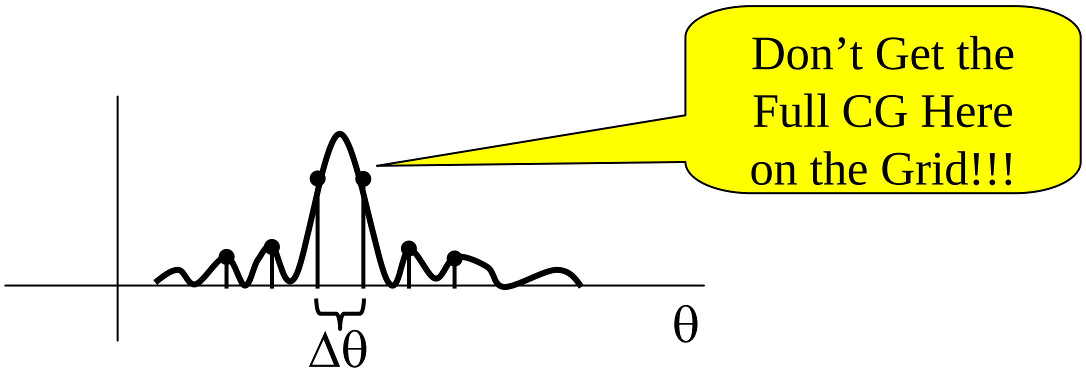
Although we did our analysis using the DTFT, the actual processing is done using the DFT.

Q: What Impact Does This Have?

Recall: DFT is DTFT computed on a grid

➔ DTFT Peak May Not Fall On the Grid

Worst Case: Peak Halfway Between Grid Points



Signal Detection Analysis (pt. 10)

Impact of Actually Using DFT rather than DTFT (cont.)

Leads to Defining "Worst-Case" Gains:

$$CG = \frac{1}{N}\left|\sum_{n=0}^{N-1} w[n]\, e^{j0.5(\Delta\theta)n}\right|$$

$$PG = \frac{\left|\sum_{n=0}^{N-1} w[n]\, e^{j0.5(\Delta\theta)n}\right|^2}{N\sum_{n=0}^{N-1} w^2[n]}$$

Num. in PG comes from CG

Don't Need to Adjust Denom b/c it accounts for the Noise Effect (which is Flat, not Peaked)

Use Worst-Case Gains: when you need to be conservative in predicting detection performance!!