Examples of <u>Finding</u> Differential Equations for Electrical Systems

For these simple circuits the trick is to first write a KVL or a KCL for the circuit.

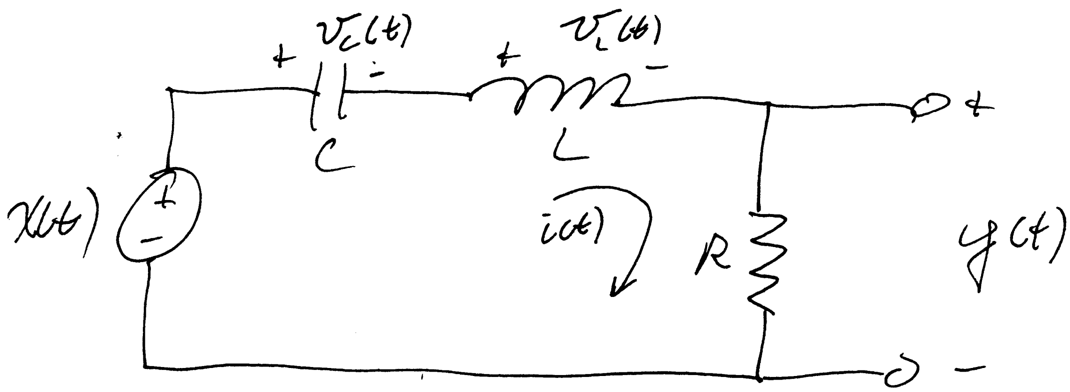
Once you have this written identify the parts of it (i.e. voltage and current variables) that **aren't** the input and output signals. You need to find a way to replace these with the input and output signals…

You do this by using:

- "device rules" (like ohms law and the I-V rules for caps and inductors)
- other rules (e.g., if you wrote a KVL first, try writing a KCL and vice versa).

Then simplify it to put it in the standard form

Example: Determine the Diff. Eq.



Write KVL around loop:

$$x(t) = v_C(t) + v_L(t) + y(t)$$

x(t) is the input so leave it
y(t) is the output so leave it (I screwed up and changed it but I fixed that later)
Now... V_C and V_L are NOT input or output variables so we need to re-write them in terms of the input and/or output!!
a good first step is to use their "device rules" that relate V-I

Should have left this as y(t)!!!

$$= \underbrace{\frac{1}{C}\int_{-\infty}^{t} i(\lambda)d\lambda}_{\text{get rid of this by differentiating both sides}} + L\frac{di(t)}{dt} + R\,i(t)$$

Then we notice that we have an integral and we can't have those in a DiffEq so we differentiate get rid of it!!!

Should have this as dy(t)/dt!!!

$$\Rightarrow \frac{dx(t)}{dt} = \boxed{\frac{1}{C}i(t) + L\frac{d^2i(t)}{dt^2}} + R\frac{di(t)}{dt}$$

These are *not* input or output variables so we need to replace them... here there is a simple connection between i(t) and y(t)

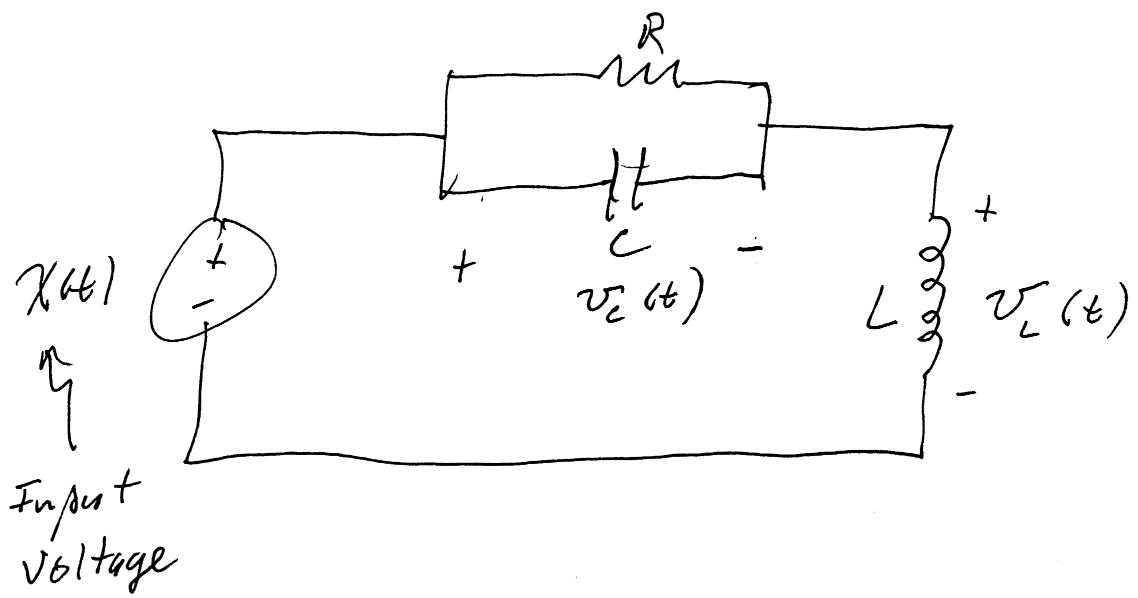
Now relate to output variable: $y(t) = i(t)R$

$\Rightarrow$ replace $i(t)$ by $y(t)/R$

$$\Rightarrow \boxed{\frac{L}{R}\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{dx(t)}{dt}}$$

Example #1

Determine the differential equation for this circuit ....



(a) ... when $v_L(t)$ is taken as the output

(b) ... when $v_c(t)$ is taken as the output

(B-1)

(a) From KVL: $v_R(t) + v_L(t) = x(t)$

or $v_R(t) = x(t) - v_L(t)$

or $i_R(t) = \dfrac{x(t) - v_L(t)}{R}$

or $v_C(t) = x(t) - v_L(t)$

because $v_C(t) = v_R(t)$ here

From KCL:

Could really start here w/ KCL:

$i_L(t) = i_R(t) + i_C(t)$

Now we use KVL and device rules to write as much as we can in terms of input & output

$i_R(t) = \dfrac{x(t) - v_L(t)}{R}$ From KVL above

$i_C(t) = C\dfrac{dv_C(t)}{dt}$ Device Rule

$i_C(t) = C\left[\dfrac{dx(t)}{dt} - \dfrac{dv_L(t)}{dt}\right]$

Now we have something that is almost completely in terms of in's & out's so we are almost there!!!

We need one more device rule...

$$i_L(t) = \underbrace{\frac{x(t) - v_L(t)}{R} + C\frac{dx(t)}{dt} - C\frac{dv_L(t)}{dt}}_{\text{in terms of in's \& outs}} \qquad (\star)$$

Must relate to In's/outs

Device Rule $\dfrac{di_L(t)}{dt} = \dfrac{1}{L}v_L(t)$

so diff. Right side of $(\star)$

We do this differentiation so we can plug this device rule in!!! When we do that we get the equation below

$$\frac{1}{L}v_L(t) = \frac{1}{R}\frac{dx(t)}{dt} - \frac{1}{R}\frac{dv_L(t)}{dt} + C\frac{d^2x(t)}{dt^2} - C\frac{d^2v_L(t)}{dt^2}$$

c

Re arrange to get:

$$\boxed{\frac{d^2 v_L(t)}{dt^2} + \frac{1}{RC}\frac{d v_L(t)}{dt} + \frac{1}{LC} v_L(t) = \frac{d^2 x(t)}{dt^2} + \frac{1}{RC}\frac{d x(t)}{dt}}$$

(b) From KVL: $v_c(t) + v_L(t) = x(t) \Rightarrow v_L(t) = x(t) - v_c(t)$

From device Rule for L:

$$v_L(t) = L\frac{di_L(t)}{dt}$$

use KCL: $i_L(t) = i_R(t) + i_c(t)$

$$= L\left[\frac{di_R(t)}{dt} + \frac{di_c(t)}{dt}\right]$$

$$\Rightarrow x(t) - v_c(t) = L\underbrace{\frac{di_R(t)}{dt}}_{i_R(t) = \frac{v_R(t)}{R} = \frac{v_c(t)}{R}} + L\underbrace{\frac{di_c(t)}{dt}}_{i_c(t) = C\frac{dv_c(t)}{dt}}$$

$$\Downarrow$$

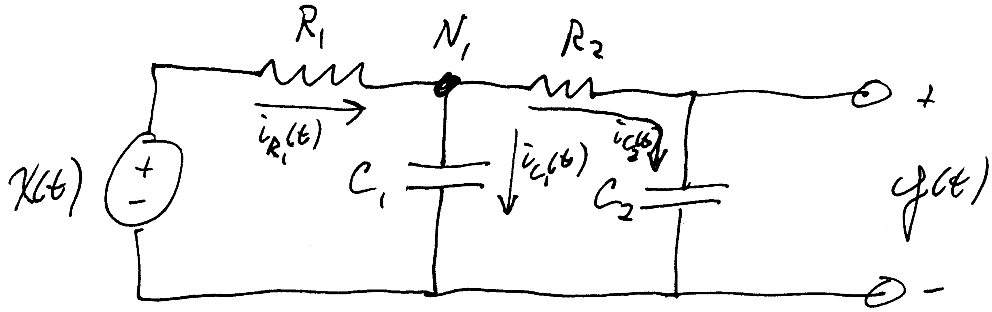
$$x(t) - v_c(t) = \frac{L}{R}\frac{dv_c(t)}{dt} + LC\frac{d^2 v_c(t)}{dt^2}$$

Re-Arrange:

$$\boxed{\frac{d^2 v_c(t)}{dt^2} + \frac{1}{RC}\frac{dv_c(t)}{dt} + \frac{1}{LC}v_c(t) = \frac{1}{LC}x(t)}$$

Example #2

Write the Diff. Eq. for the following "2-stage" RC circuit:



<u>Node Eq. at N_1:</u> (Node Equations are a form of KCL)

$$\underbrace{i_{R_1}(t)}_{\downarrow} = \underbrace{i_{C_1}(t)}_{\downarrow} + i_{C_2}(t)$$

$$\frac{x(t) - v_{C_1}(t)}{R_1} = C_1 \frac{dv_{C_1}(t)}{dt} + C_2 \frac{dy(t)}{dt} \qquad (\star)$$

Almost have what we want... need to get $v_{C_1}(t)$ out of the equation!

<u>KVL around 2nd Mesh</u>

$$v_{C_1}(t) = \underbrace{v_{R_2}(t)}_{} + y(t)$$

$$= R_2 i_{C_2}(t) = R_2 \left[C_2 \frac{dy(t)}{dt} \right]$$

$(\star\star)$ Use all this to eliminate $v_{C_1}(t)$

Put (☆☆) into (☆) to get: (E)

First re-write (☆):

$$C_2 \frac{dy(t)}{dt} + C_1 \frac{dv_{c_1}(t)}{dt} + \frac{1}{R_1} v_{c_1}(t) = \frac{1}{R_1} x(t)$$

From (☆☆) we get:

$$v_{c_1}(t) = R_2 C_2 \frac{dy(t)}{dt} + y(t)$$

and

$$\frac{dv_{c_1}(t)}{dt} = R_2 C_2 \frac{d^2 y(t)}{dt^2} + \frac{dy(t)}{dt}$$

We get:

$$C_1 R_2 C_2 \frac{d^2 y(t)}{dt^2} + \left(C_1 + \frac{R_2 C_2}{R_1} + C_2\right)\frac{dy(t)}{dt} + \frac{1}{R_1} y(t) = \frac{1}{R_1} x(t)$$

which could then be normalized by multiplying by $1/C_1 R_2 C_2$

DC Motor Example ("Armature Control")

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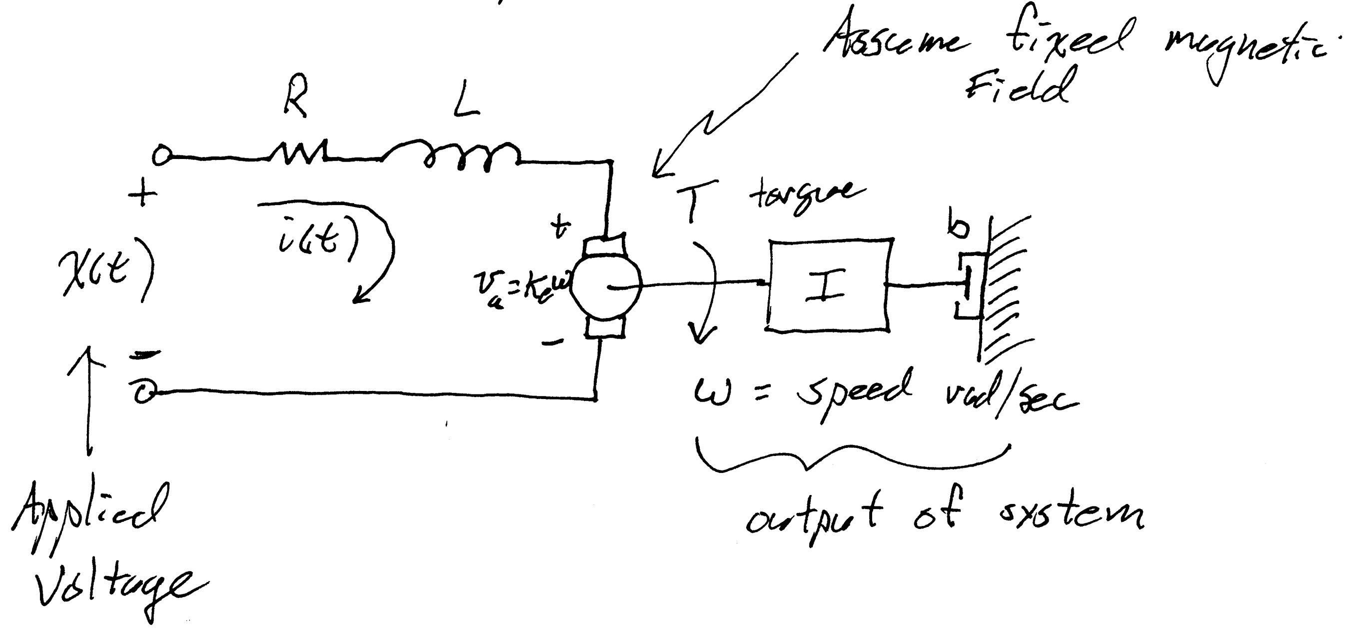

- DC Motor consists of lots of wire ~~wound~~ wound on an "armature" $\Rightarrow$ motor has "parasitic" R & L

- As motor armature turns, the coils cut magnetic flux lines $\Rightarrow$ induces voltage proportional to speed

$$v_a(t) = K_e\,\omega(t)$$

Now write KVL around the armature circuit:

$$x(t) = R\,i(t) + L\frac{di(t)}{dt} + K_e\,\omega(t) \qquad (*)$$

- The current $i(t)$ in the windings interacts w/ the magnetic field to create a force which provides the motor torque $T(t)$

$$T(t) = K_t\,i(t) \qquad (**)$$

Finally, rotational mechanics provides

$$T(t) = \underbrace{I \frac{d\omega(t)}{dt}}_{\substack{\text{Torque} \\ \text{due to} \\ \text{mass } I}} + \underbrace{b\,\omega(t)}_{\substack{\text{Torque} \\ \text{due to} \\ \text{friction}}} \qquad (***)$$

Putting (***) into (**) and dividing by K_t gives:

$$i(t) = \frac{I}{K_t}\frac{d\omega(t)}{dt} + \frac{b}{K_t}\,\omega(t)$$

Now put this in (*) to get:

$$\boxed{\frac{LI}{K_t}\frac{d^2\omega(t)}{dt} + \left(\frac{RI + Lb}{K_t}\right)\frac{d\omega(t)}{dt} + \left(K_e + \frac{Rb}{K_t}\right)\omega(t) = x(t)}$$

$\Rightarrow$ Second-Order Diff. Eq.

$\Rightarrow$ Could be Oscillatory!