Binghamton University
State University of New York

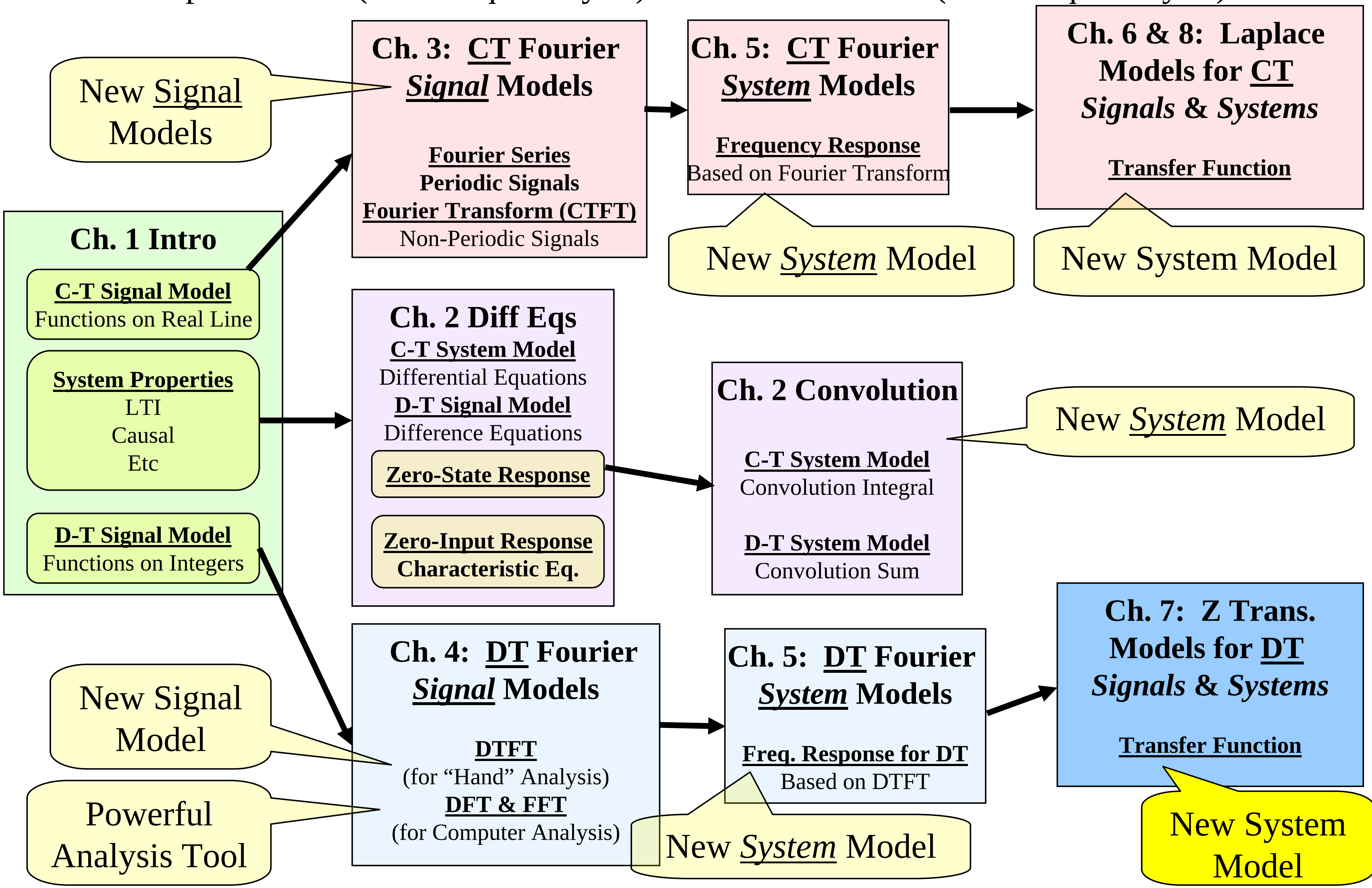
EECE 301
Signals & Systems
Prof. Mark Fowler

Note Set #33

- D-T Systems: Z-Transform … "Power Tool" for system analysis
- Reading Assignment: Sections 7.1 – 7.3 of Kamen and Heck

Course Flow Diagram

The arrows here show conceptual flow between ideas. Note the parallel structure between the pink blocks (C-T Freq. Analysis) and the blue blocks (D-T Freq. Analysis).



Ch. 11 Z-Transform & D-T Systems

Z-Transform does for DT systems what the Laplace Transform does for CT systems

Z-T is used to

Solve difference equations with initial conditions	Solve zero-state systems using the transfer function

We will:

- Define the ZT
- See its properties
- Use the ZT and its properties to analyze D-T systems

Section 7.1 Z-transform definitions

Given a D-T signal $x[n]$ $-\infty < n < \infty$ we've already seen how to use the DTFT:

$$DTFT: X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

Periodic in Ω with period 2π

<ins>Recall</ins>: For C-T case, the FT doesn't converge for some signals… the LT mitigates this problem by including decay in the transform

$$e^{-j\omega t} \quad vs. \quad e^{-(\sigma + j\omega)t} \equiv e^{-st}$$

Controls decay of integrand

So, for D-T signals we include decay into the transform; but in a slightly different way:

$$e^{-j\Omega n} \quad vs. \quad \alpha^{-n} e^{-j\Omega n} \equiv (\alpha e^{j\Omega})^{-n} \equiv z^{-n}$$

Controls decay of summand

So for the Laplace transform we looked at: $s = \sigma + j\omega$ which is in rect. form

But, for Z-transform we use: $z = \alpha e^{j\Omega}$ which is in polar form

Q: Why the change?

A: Suffice to say…it has to do with the periodic nature of the DTFT.

Remember that the DTFT is a periodic function of Ω… and by using $z = \alpha e^{j\Omega}$ we stick Ω in as an angle which forces the periodic dependence on Ω.

Just like for Laplace… there are two forms of the Z-Transform:

Two sided Z-transform

$$X_2(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \qquad z \text{ is complex-valued}$$

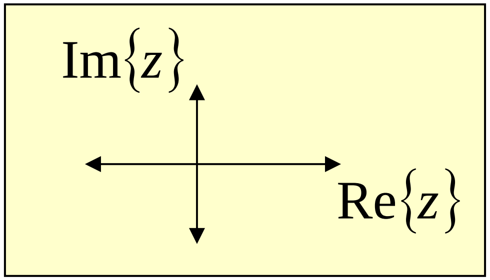
One sided Z-transform

$$X_1(z) = \sum_{n=0}^{\infty} x[n] z^{-n} \qquad z \text{ is complex-valued}$$

If $x[n]$ is a causal signal: $X_1(z) = X_2(z)$

Our Focus is Here

So… the Z-Transform gives a complex-valued function on the "z-plane"



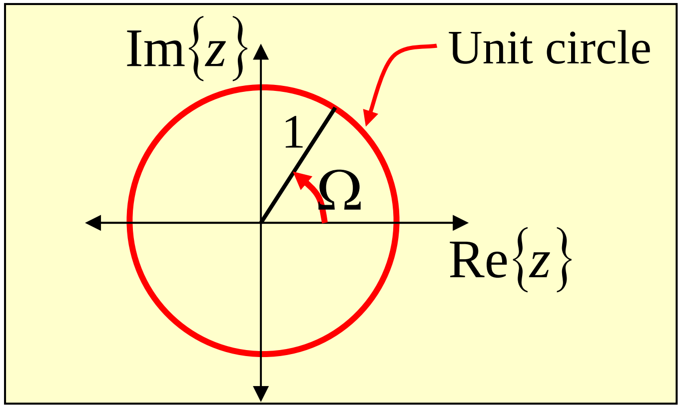
Recall: for Laplace we had the *s*-plane… and we divided it into two parts:

- those values of *s* to the left of the *j*ω-axis (left-half plane)
- those values of *s* to the right of the *j*ω-axis (right-half plane)

"**Unit Circle**" = all *z* such that |z| = 1, i.e. all $z = e^{j\Omega}$

For the Z-Transform we'll need to divide the plane into two parts:

- those values of *z* inside the unit circle
- those values of *z* outside the unit circle



Region of Convergence (ROC)

Set of all *z* values for which the sum in the ZT definition converges

Each signal has its own region of convergence.

(Same idea as for Laplace Transform)

Example of Finding the ZT: Unit Impulse Sequence

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

$$Z\{\delta[n]\} = \sum_{n=0}^{\infty} \delta[n] z^{-n}$$

$$= 1 \times z^{0} + 0 \times z^{-1} + 0 \times z^{-2} + \cdots$$

$$= 1$$

$$\delta[n] \quad \leftrightarrow \quad 1$$

ROC = all complex #'s

This result and many others are on Table of Z Transforms available on my website… please use it rather than the one in your book, which has some errors

Example of Finding the ZT: Unit Step $u[n]$

$$U(z) = \sum_{n=0}^{\infty} u[n] z^{-n} = \sum_{n=0}^{\infty} z^{-n} = \frac{z}{z-1} = \frac{1}{1-z^{-1}}$$

ROC = all z such that $|z| > 1$

Using standard result for "geometric sum"

$$u[n] \leftrightarrow \frac{z}{z-1} = \frac{1}{1-z^{-1}}$$

Example of Finding the ZT: Causal Exponential

$$x[n] = a^n u[n]$$

Again using geometric sum: $X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n = \frac{z}{z-a} = \frac{1}{1-az^{-1}}$

ROC = all z such that $|z| > |a|$

$$a^n u[n] \leftrightarrow \frac{z}{z-a} = \frac{1}{1-az^{-1}}$$

Relationship between ZT & DTFT

Recall: for some signals the CTFT was embedded in the LT

(If the ROC includes the $j\omega$-axis)

We have a similar condition for the DTFT and the ZT…

If ROC includes the unit circle, then we can say that: $$X(\Omega) = X(z)\Big|_{z=e^{j\Omega}}$$

$X(\Omega)$ = "walk around the unit circle" and get $X(z)$ values

Explains why $X(\Omega)$ is periodic… Ω is an "angle around the unit circle"

$\Rightarrow$ Once we've walked around the unit circle… going farther just repeats the values $X(z)$ that we are grabbing

$\Rightarrow$ We only need to worry about $\Omega \in$ [-π to π)

7.3 Inverse Z-T

Same story as for LT: using the integral inversion formula is hard!

$\Rightarrow$ Use partial fractions

The use of partial fractions here is *almost* exactly the same as for Laplace transforms…

… the only difference is that you first divide by z ***before*** performing the partial fraction expansion… then after expanding you multiply by z to get the final expansion.

Example of Partial Fraction for Inverse ZT:

Suppose you want to find the inverse ZT of

$$Y(z) = \frac{z+1}{z^2 + \frac{3}{4}z + \frac{1}{8}}$$

First divide $Y(z)$ by z to get:

$$\frac{Y(z)}{z} = \frac{z+1}{z^3 + \frac{3}{4}z^2 + \frac{1}{8}z}$$

Then use matlab's residue to do a partial fraction expansion on $Y(z)/z$

```
[r,p,k]=residue([1 1],[1 0.75 0.125 0])

r =                p =                k = []
    4                 -0.5000
  -12                 -0.2500
    8                  0
```

Then we have:

$$\frac{Y(z)}{z} = \frac{4}{z+\frac{1}{2}} - \frac{12}{z+\frac{1}{4}} + \frac{8}{z}$$

$$\Rightarrow \quad Y(z) = \frac{4z}{z+\frac{1}{2}} - \frac{12z}{z+\frac{1}{4}} + 8$$

Now… the point of dividing by z becomes clear… you get terms like this (with z's in the numerator)… and they are on the ZT table!!!

$$\Rightarrow \quad y[n] = 4(-\tfrac{1}{2})^n u[n] - 12(-\tfrac{1}{4})^n u[n] + 8\delta[n]$$

11.2 Properties of ZT

Linearity: Same ideas as for CTFT, DTFT, and LT

Right Shift for Causal Signal

Let $x[n] = 0, \ n < 0$

$$\text{If } \ x[n] \leftrightarrow X(z), \quad \text{then} \quad x[n-q] \leftrightarrow z^{-q} X(z)$$

$$\text{"Proof"}: \quad X(z) = x[0]z^{0} + x[1]z^{-1} + x[2]z^{-2} + \ldots$$

$$Z\{x[n-q]\} = \underbrace{0z^{0} + 0z^{-1} + \ldots + 0z^{-q+1}}_{=0} + x[0]z^{-q} + x[1]z^{-q-1} + \ldots$$

$$= x[0]z^{0}z^{-q} + x[1]z^{-1}z^{-q} + x[2]z^{-2}z^{-q} + \ldots$$

Pull out the z^{-q}

$$= z^{-q}\underbrace{\left[x[0]z^{0} + x[1]z^{-1} + \ldots\right]}_{= X(z)}$$

Example of Applying the Right-Shift Property for Causal Signals

Suppose we want to find the Z-T of the pulse signal:

$$p[n] = \begin{cases} 1, \; n = 0, 1, 2, ..., \; q-1 \\ 0, \; else \end{cases}$$

Well.. We can write this pulse in terms of the unit step:

$$p[n] = u[n] - u[n-q]$$

Now, by linearity of the ZT we have: $P(z) = Z\{u[n]\} - Z\{u[n-q]\}$

But we already know that $Z\{u[n]\} = \dfrac{z}{z-1}$

Using the Right-Shift Property gives $Z\{u[n-q]\} = z^{-q}\dfrac{z}{z-1}$

So…

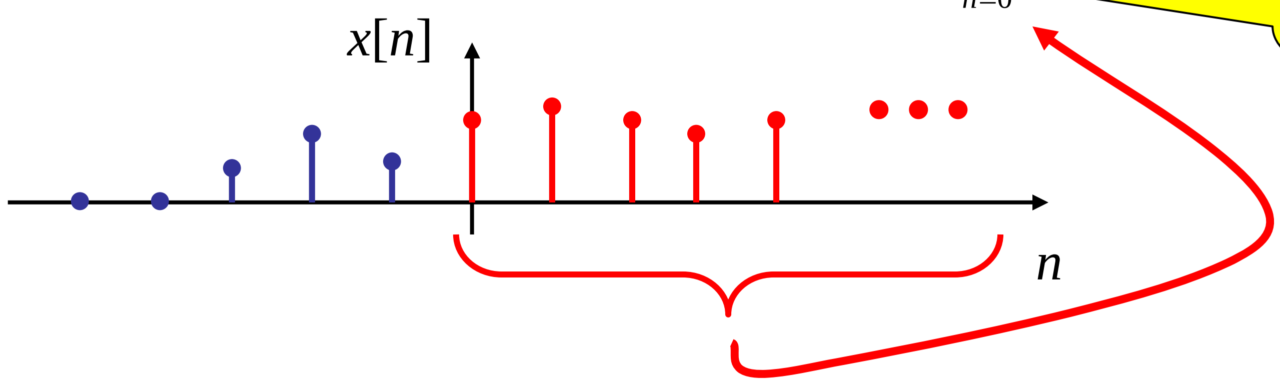
$$P(z) = \left[\frac{z}{z-1}\right] - z^{-q}\left[\frac{z}{z-1}\right] = \frac{z(1-z^{-q})}{z-1}$$

One-Sided ZT of the Right shift of Non-causal x[n]

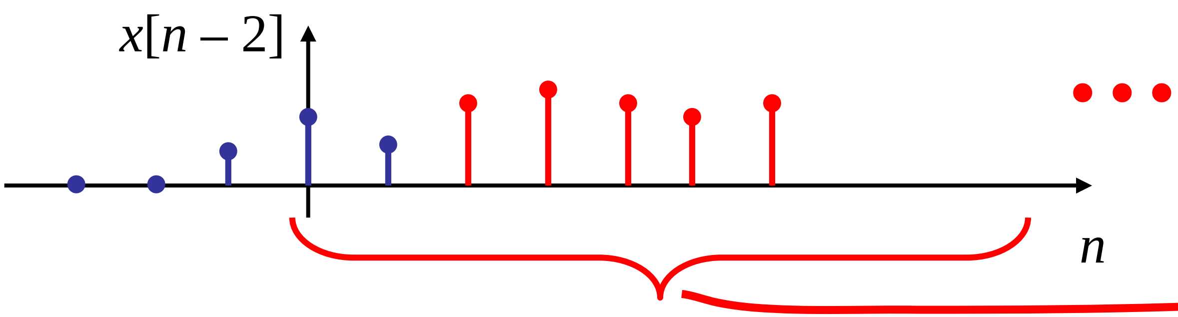
Let $x[n]$ be a non-causal signal… $x[n] \neq 0$ for some $n < 0$

Then the One-Sided ZT is: $x[n] \leftrightarrow X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$

Because this is the One-Sided ZT… not all non-zero values of $x[n]$ are used here!!!



Note that right-shifting a non-causal signal brings new values into the one-sided ZT summation!!!



$Z\{x[n-2]\} = \sum_{n=0}^{\infty} x[n-2] z^{-n}$

What is $Z\{x[n-q]\}$ in terms of $X(z)$??

We'll write this property for the first 2 values of q…

$$x[n-1] \quad \leftrightarrow \quad z^{-1}X(z) + x[-1]$$

$$x[n-2] \quad \leftrightarrow \quad z^{-2}X(z) + x[-1]z^{-1} + x[-2]$$

$$\vdots \qquad\qquad \vdots$$

… and then write the general result:

$$x[n-q] \quad \leftrightarrow \quad z^{-q}X(z) + x[-1]z^{-q+1} + x[-2]z^{-q+2} + \ldots + z^{-1}x[-q+1] + x[-q]$$

"Proof" for $q = 2$

$$Z\{x[n-q]\} = x[-2]z^{0} + x[-1]z^{-1} \quad + \quad x[0]z^{-2} + x[1]z^{-3} + \ldots$$

$$= \underbrace{x[-2]z^{0} + x[-1]z^{-1}}_{} \quad + \quad z^{-2}\underbrace{\left(x[0]z^{0} + x[1]z^{-1} + \ldots\right)}_{X(z)}$$

Parts that get "shifted into" the one-sided ZT's "machinery"

Convolution Property

For two causal signals $x[n]$ & $h[n]$ with one-sided ZTs $X(z)$ & $H(z)$

… we have:

$$x[n]*h[n] \quad \leftrightarrow \quad X(z)H(z)$$

Just like for CTFT, LT, & DTFT…

…Convolution Transforms to Multiplication!!!

There are several other properties… they are listed on the Table of Z Transform Properties on my Webpage… please use that table rather than the one in the book, which has some errors.