Partial Fraction Expansion

When trying to find the inverse Laplace transform (or inverse z transform) it is helpful to be able to break a complicated ratio of two polynomials into forms that are on the Laplace Transform (or z transform) table. We will illustrate here using Laplace transforms. This can be done using the method of "partial fraction expansion" (PFE), which is the reverse of finding a common denominator and combining fractions. It is possible to do PFE by hand or it is possible to use MATLAB to help. We will illustrate hand computation only for the simplest case when there are no repeated roots and the order of the numerator polynomial is strictly less than the order of the denominator polynomial.

First we will show **why** the "by-hand" method works and then we will show **how** one actually does it. Then we will illustrate how to use MATLAB to do PFE.

Why It Works
For that case, suppose we have a LT $Y(s)$ that we wish to invert but it is not on our table. Suppose that the denominator of $Y(s)$ can be factored as

$$Y(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{d_N s^N + d_{N-1} s^{N-1} + \cdots + d_1 s + d_0}$$

$$= \frac{N(s)}{d_N (s - p_1)(s - p_2)\cdots(s - p_N)}$$

For the case we are considering this can be expanded into the form

$$Y(s) = \frac{r_1}{(s - p_1)} + \frac{r_2}{(s - p_2)} + \cdots + \frac{r_N}{(s - p_N)}$$

where the r_i are numbers called the "residues" of the expansion.

The goal of doing a PFE is to find the residues so you can form the right-hand side of the above equation.

So… we need a way to solve this for each r_i. Let's see how to do that for r_1 (the other r_i's are done the same). We multiply this equation on both sides by $(s - p_1)$ and get

$$(s - p_1)Y(s) = \frac{r_1(s - p_1)}{(s - p_1)} + \frac{r_2(s - p_1)}{(s - p_2)} + \cdots + \frac{r_N(s - p_1)}{(s - p_N)}$$

$$= r_1 + \frac{r_2(s - p_1)}{(s - p_2)} + \cdots + \frac{r_N(s - p_1)}{(s - p_N)}$$

where in the second line we have canceled the common $(s - p_1)$ in the first term on the right hand side (…. this is the whole point of the process!! We have isolated r_1 !! But there are still a bunch of terms left in the way… but we can now get rid of them as follows.).

Now note that if we let $s = p_1$ then all the terms containing the *other* residues disappear:

$$[(s - p_1)Y(s)]_{s=p_1} = r_1 + \frac{r_2(p_1 - p_1)}{(s - p_2)} + \cdots + \frac{r_N(p_1 - p_1)}{(s - p_N)} = r_1$$

Now… you might wonder why this doesn't make the left-hand side disappear since there is now also a $(p_1 - p_1)$ term on that side!!! Remember that there is an $(s - p_1)$ term in the denominator of $Y(s)$ that cancels the multiplied $(s - p_1)$ term!!!

So this gives us a nice mathematical result on how to find the residues:

$$r_i = [(s - p_i)Y(s)]_{s=p_i}$$

Remember that this equation only hold for the case when there are no repeated roots and the order of the numerator of $Y(s)$ is strictly less than the order of the denominator of $Y(s)$

How to Do It
Illustrate with an example. Find the inverse LT of

$$Y(s) = \frac{3s - 1}{s^2 + 3s + 2}$$

First we have to factor the denominator. Using the quadratic formula we get the roots and the write the factored form as

$$Y(s) = \frac{3s - 1}{(s + 2)(s + 1)}$$

Then we use the equation we derived above:

$$r_1 = \left[(s+2)Y(s)\right]_{s=-2} = \left[\frac{(s+2)(3s-1)}{(s+2)(s+1)}\right]_{s=-2}$$

$$= \left[\frac{(3s-1)}{(s+1)}\right]_{s=-2}$$

or finally $r_1 = 7$.

Notice that we **_MUST_** cancel the multiplied factor **_BEFORE_** we evaluate at the root value.

Similarly,

$$r_2 = \left[(s+1)Y(s)\right]_{s=-1} = \left[\frac{(s+1)(3s-1)}{(s+2)(s+1)}\right]_{s=-1}$$

$$= \left[\frac{(3s-1)}{(s+2)}\right]_{s=-1}$$

or finally $r_2 = -4$.

Partial Fraction Expansion via MATLAB

The **"residue" function of MATLAB can be used to compute the partial fraction expansion (PFE)** of a ratio of two polynomials.

This can be used for **Laplace transforms** or **Z transforms**, although we will illustrate it with Laplace transforms here.

The "residue" command gives three pieces of information:
> ‣ the residues are given in output vector r,
> ‣ the poles are given in output vector p,
> ‣ the so-called direct terms are given in output vector k.

When the order of the numerator polynomial:
> ‣ is less than the order of the denominator polynomial there will be no direct terms.
>
> ‣ equals the order of the denominator there will be one direct term;.
>
> ‣ is one greater than the order of the denominator there will be two direct terms.
>
> ‣ etc.

That is, **when the order of the numerator is p greater than the order of the denominator (with p≥0) there will be p+1 direct terms**.

The "residue" command requires two input vectors:
> ‣ one holding the coefficients of the numerator and
> ‣ one holding the coefficients of the numerator.

The right-most element in these vectors corresponds to the s^0 coefficient, the next element to the left is the s^1 coefficient, etc., until you reach the highest power; if a power is not present it has a zero coefficient.

It is easiest to explain how to use "residue" by giving examples.

Ex. #1: **No Direct Terms**, **No Repeated Roots**, **No Complex Roots**

$$Y(s) = \frac{3s - 1}{s^2 + 3s + 2}$$

In this case there will be no direct terms because the numerator order is lower than the denominator order.

The **numerator vector** is [3   −1]

The **denominator vector** is [1   3   2].

Note: If there is a power "missing" here is how you handle that… say the denominator were $s^2 + 2$ then the den vector is [1 0 2]

The **command and its result** is:

» [r,p,k]=residue([3 -1],[1 3 2])

r =

   7
  -4

p =

  -2
  -1

k =
   []

Note that k is empty (denoted by []) showing that there are no direct terms.

The **numbers in r are the residues** – they are the numbers that go in the numerator of each term in the expansion.

The **numbers in p are the poles** – they are the numbers subtracted from *s* to form the denominators of the terms.

We have a **pole at *s* = -2 with a residue of 7** and….
        …a **pole at *s* = -1 with a residue of –4**; thus:

$$Y(s) = \frac{3s - 1}{s^2 + 3s + 2} = \frac{7}{s + 2} + \frac{-4}{s + 1}$$

and then using the Laplace Transform Table for each of the two terms in the expansion we get:

$$y(t) = 7e^{-2t}u(t) - 4e^{-t}u(t)$$

Ex. #2: One Direct Term, No Repeated Roots, No Complex Roots

$$Y(s) = \frac{2s^2 + 3s - 1}{s^2 + 3s + 2}$$

» [r,p,k]=residue([2 3 -1],[1 3 2])

r =
  -1
  -2

p =
  -2
  -1

k =
  2

Thus, we have: **a pole at**: -2 w/ residue -1
                          a pole at: -1 w/ residue -2;

…. But in this case we **also have a single direct term of 2**, which just adds a term of 2 to the expansion:

$$Y(s) = \frac{2s^2 + 3s - 1}{s^2 + 3s + 2} = \frac{-1}{s+2} + \frac{-2}{s+1} + 2$$

and then using the Laplace Transform Table for each of the three terms in the expansion we get:

$$y(t) = -e^{-2t}u(t) - 2e^{-t}u(t) + 2\delta(t)$$

Ex. #3: Two Direct Terms, No Repeated Roots, No Complex Roots

$$H(s) = \frac{7s^3 + 2s^2 + 3s - 1}{s^2 + 3s + 2}$$

» [r,p,k]=residue([7 2 3 -1],[1 3 2])

r =

   55
   -9

p =

   -2
   -1

k =

   7   -19

This **gives two terms in the vector k** so we know we **have two direct terms: 7s and -19**.

(**Note**: **if there were M elements in vector k** the left most term would be the coefficient of a z^M order direct term.) Thus, we have

$$Y(s) = \frac{7s^3 + 2s^2 + 3s - 1}{s^2 + 3s + 2} = \frac{55}{s+2} + \frac{-9}{s+1} + 7s - 19$$

Get a derivative of the delta from the differentiation property

and using the Laplace Transform Table we get:

$$y(t) = 55e^{-2t}u(t) - 9e^{-t}u(t) + 7\dot{\delta}(t) - 19\delta(t)$$

Ex. #4: **No Direct Terms, A Double Root, No Complex Roots**

$$Y(s) = \frac{2s^2 + 3s - 1}{s^3 + 5s^2 + 8s + 4} = \frac{2s^2 + 3s - 1}{(s+1)(s+2)^2}$$

» [r,p,k]=residue([2 3 -1],[1 5 8 4])

r =

   4.0000
  -1.0000
  -2.0000

p =

  -2.0000
  -2.0000
  -1.0000

k =

   []

Note that **p has two elements listed as -2.0000**: that means that there is a **double root**.

Note also that there are **two residues for these poles** – namely, 4 and -1:
> ‣ **First residue** is for the **first-order term for the double root**
> ‣ **Second residue** is for the **second-order term for the double root**.

Thus, the expansion is

$$Y(s) = \frac{2s^2 + 3s - 1}{s^3 + 5s^2 + 8s + 4} = \frac{2s^2 + 3s - 1}{(s+1)(s+2)^2} = \frac{4}{s+2} + \frac{-1}{(s+2)^2} + \frac{-2}{s+1}$$

and using the Laplace Transform Table we get:

$$y(t) = 4e^{-2t}u(t) - te^{-2t}u(t) - 2e^{-t}u(t)$$

Note, in general: if there were an **M$^{\text{th}}$ order pole** we would have **a term in the expansion of $Y(s)$ for each order up to and including M** for that pole.

Ex. #5: **No Direct Terms**, **No Repeated Roots**, **Two Complex Roots**

$$Y(s) = \frac{3s - 1}{s^2 + 5s^2 + 9s + 5}$$

$$= \frac{3s - 1}{(s + 1)(s + (2 + j))(s + (2 - j))}$$

This shows how the given **denominator polynomial** is **factored into one real pole and two complex poles**.

Note that the **two complex poles are conjugates of each other** – this is a necessary condition for the coefficients of the polynomial to be real valued.

Thus, for the cases we are interested in, **we will *always* have complex poles occur in "conjugate pairs"**.

The **use of "residue" is no different here**,
.........but **what we do with the output is a little different**.

» [r,p,k]=residue([3 -1],[1 5 9 5])

r =
   1.0000 - 2.5000i
   1.0000 + 2.5000i
  -2.0000

p =
  -2.0000 + 1.0000i
  -2.0000 - 1.0000i
  -1.0000

k = []

$$Y(s) = \frac{3s-1}{s^2 + 5s^2 + 9z + 5} = \frac{1-j2.5}{(s+(2-j))} + \frac{1+j2.5}{(s+(2+j))} + \frac{-2}{(s+1)}$$

Now we invert each of these terms using the <u>same</u> LT pair on the LT table using the inverse LT for $1/(s+b)$ where b is real for the real pole and is complex for the complex poles. So

$$y(t) = (1-j2.5)e^{(-2+j)t}u(t) + (1+j2.5)e^{(-2-j)t}u(t) - 2e^{-t}u(t)$$

$$= (2.69e^{-j1.19})e^{(-2+j)t}u(t) + (2.69e^{+j1.19})e^{(-2-j)t}u(t) - 2e^{-t}u($$

$$= 2.69e^{-2t}\left[e^{j(t-1.19)} + e^{-j(t-1.19)}\right]u(t) - 2e^{-t}u(t)$$

$$= 5.38e^{-2t}\cos(t-1.19)u(t) - 2e^{-t}u(t)$$

The 1<u>st</u> line in the above equation: comes from using the table entry.

The 2<u>nd</u> line: converts the (1±j2.5) terms into polar form

The 3<u>rd</u> line: pulls out the common $2.69e^{-2t}$ term and combines the complex exponential terms

The 4<u>th</u> line: Uses Euler's formula to combine the two complex sinusoids into a real sinusoid (in this case with a frequency of 1 rad/sec and a phase of –1.19 rad).