Binghamton University
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EECE 301
Signals & Systems
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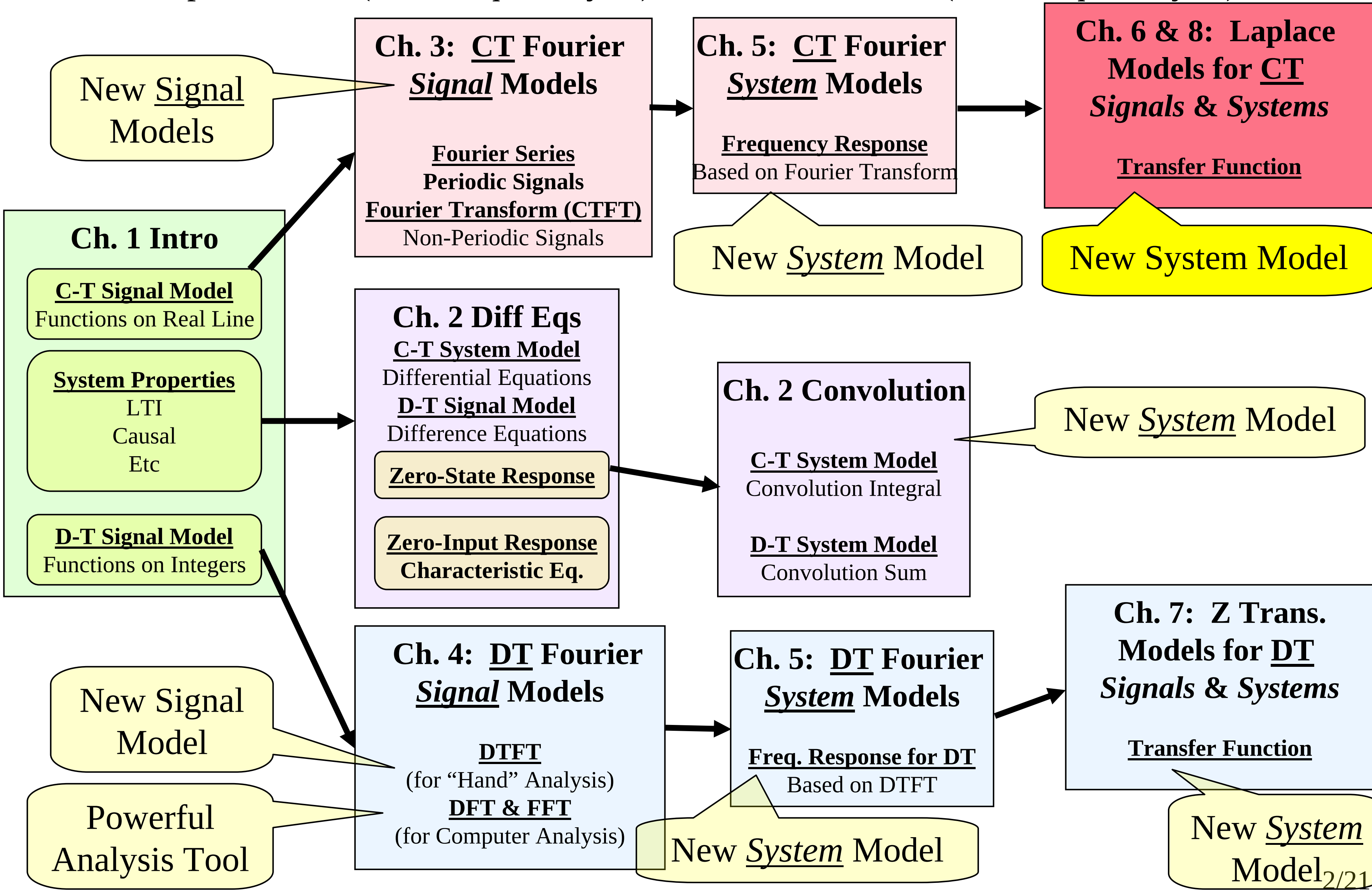
Note Set #28

- C-T Systems: Laplace Transform… Solving Differential Equations
- Reading Assignment: Section 6.4 of Kamen and Heck

Course Flow Diagram

The arrows here show conceptual flow between ideas. Note the parallel structure between the pink blocks (C-T Freq. Analysis) and the blue blocks (D-T Freq. Analysis).



6.4 Using LT to solve Differential Equations

In Ch. 2 we saw that the solution to a linear differential equation has two parts:

$$y_{total}(t) = \underbrace{y_{zs}(t)}_{} + \underbrace{y_{zi}(t)}_{}$$

We've seen how to find this using:
"convolution w/ impulse response" (Ch. 2)

or using

"multiplication w/ frequency response" (Ch. 5)

We've seen how to find this using the characteristic equation, its roots, and the so-called "characteristic modes" (Ch. 2)

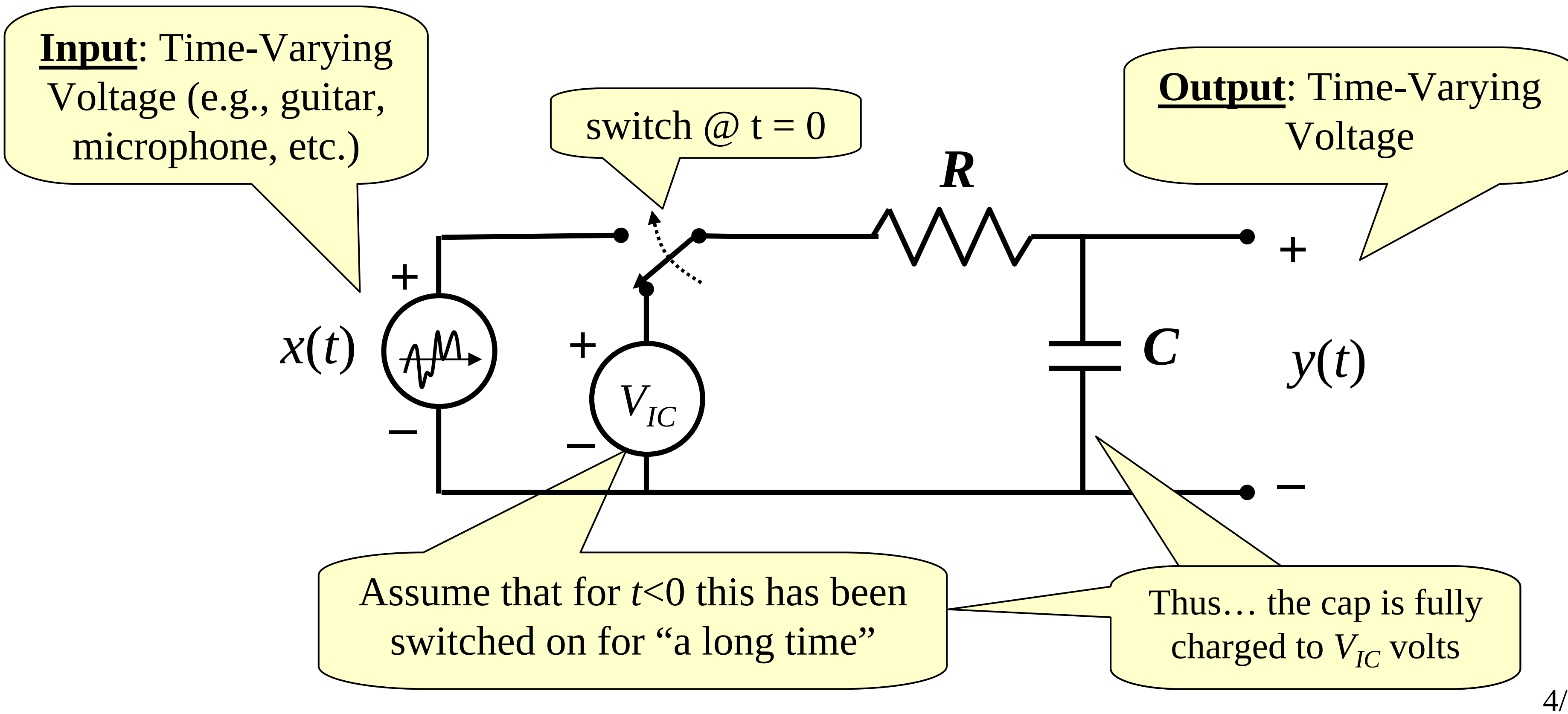
Here we'll see how to get $y_{total}(t)$ using LT…
… get both parts with one tool!!!

First-order case: Let's see this for a **1st-order Diff. Eq.** with a **causal input** and a **non-zero initial condition** just before the causal input is applied.

The 1st-order Diff. Eq. describes: a simple RC or RL circuit.

The causal input means: we switch on some input at time $t = 0$.

The initial condition means: just before we switch on the input the capacitor has a specified voltage on it (i.e., it holds some charge).



This circuit is then described by this Diff. Eq.:

$$\frac{dy(t)}{dt} + \frac{1}{RC} y(t) = \frac{1}{RC} x(t)$$

$x(t) = 0, \; t < 0$

Cap voltage… just before $x(t)$ "turns on" → With IC $y(0^-) = V_{IC}$

For this ex. we'll solve the <u>general</u> 1st-order Diff. Eq.:

$$\frac{dy(t)}{dt} + ay(t) = bx(t)$$

Now the **<u>key steps</u>** in using the LT are:

- take the LT of both sides of the <u>Differential Equation</u>…
- use the LT properties where appropriate…
- solve the resulting <u>Algebraic Equation</u> for $Y(s)$
- find the inverse LT of the resulting $Y(s)$

Laplace Transform:

Differential Equation… (Hard to solve)

turns into an…

Algebraic Equation (Easy to solve)

We now apply these steps to the 1st-order Diff. Eq.:

$$\mathcal{L}\left\{\frac{dy(t)}{dt} + ay(t)\right\} = \mathcal{L}\{bx(t)\}$$

Apply LT to both sides

$$\mathcal{L}\left\{\frac{dy(t)}{dt}\right\} + a\mathcal{L}\{y(t)\} = b\mathcal{L}\{x(t)\}$$

Use Linearity of LT

$$\left[sY(s) - y(0^-)\right] + aY(s) = bX(s)$$

Use Property for LT of Derivative… accounting for the IC

$$Y(s) = \frac{y(0^-)}{s+a} + \frac{b}{s+a}X(s)$$

Solve algebraic equation for $Y(s)$

Part of sol'n driven by IC
"Zero-Input Sol'n"

Part of sol'n driven by input
"Zero-State Sol'n"

Now… the "hard" part is to find the inverse LT of $Y(s)$

Note that 1/(s+a) plays a role in both parts…

Hey! s+a is the Characteristic Polynomial!!

Example: RC Circuit

Now we apply these general ideas to solving for the output of the previous RC circuit with a unit step input…. $x(t)=u(t)$

$$\frac{dy(t)}{dt}+\frac{1}{RC}y(t)=\frac{1}{RC}x(t) \quad \Rightarrow \quad Y(s)=\frac{y(0^-)}{s+1/RC}+\left[\frac{1/RC}{s+1/RC}\right]X(s)$$

This "transfers" the input $X(s)$ to the output $Y(s)$

We'll see this later as "The Transfer Function"

Now… we need the LT of the input…

From the LT table we have: $x(t)=u(t) \quad \leftrightarrow \quad X(s)=\frac{1}{s}$

$$Y(s)=\frac{y(0^-)}{s+1/RC}+\left[\frac{1/RC}{(s+1/RC)}\right]\frac{1}{s}$$

Now we have "just a function of s" to which we apply the ILT…

So now applying the ILT we have:

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{y(0^-)}{s+1/RC} + \left[\frac{1/RC}{(s+1/RC)s}\right]\right\}$$

Apply LT to both sides

$$y(t) = \mathcal{L}^{-1}\left\{\frac{y(0^-)}{s+1/RC}\right\} + \mathcal{L}^{-1}\left\{\left[\frac{1/RC}{(s+1/RC)s}\right]\right\}$$

Linearity of LT

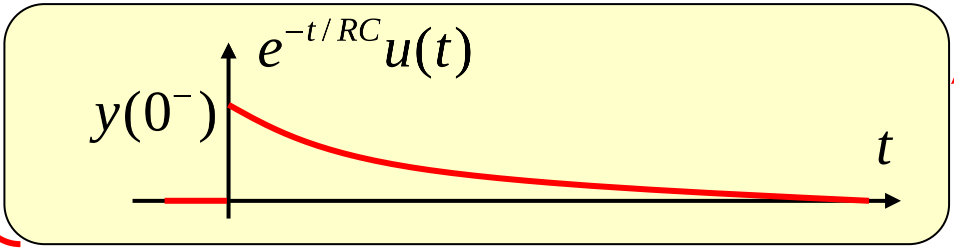
This part (zero-input sol'n) is easy…

Just look it up on the LT Table!!

This part (zero-state sol'n) is harder…

It is **NOT** on the LT Table!!

$$\mathcal{L}^{-1}\left\{\frac{y(0^-)}{s+1/RC}\right\} = y(0^-)e^{-(t/RC)}u(t)$$



So… the part of the sol'n due to the IC (zero-input sol'n) decays down from the IC voltage

Now let's find the other part of the solution… the zero-state sol'n… the part that is driven by the input:

$$y(t) = \mathcal{L}^{-1}\left\{\frac{y(0^-)}{s+1/RC}\right\} + \mathcal{L}^{-1}\left\{\left[\frac{1/RC}{(s+1/RC)s}\right]\right\}$$

We can *factor* this function of s as follows:

$$\mathcal{L}^{-1}\left\{\left[\frac{1/RC}{(s+1/RC)s}\right]\right\} = \mathcal{L}^{-1}\left\{\left[\frac{1}{s} - \frac{1}{s+1/RC}\right]\right\}$$

Can do this with "Partial Fraction Expansion", which is just a "fool-proof" way to factor

$$= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1/RC}\right\}$$

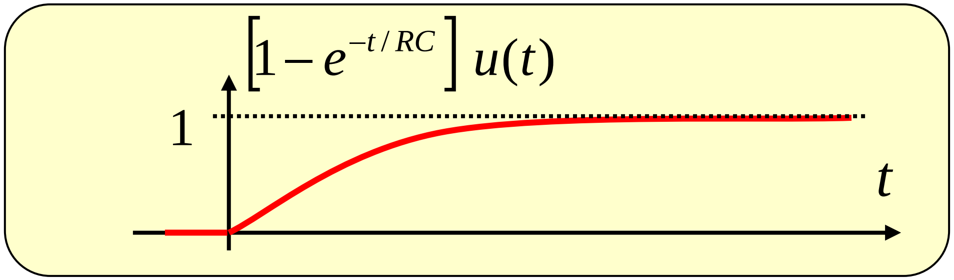
Linearity of LT

Now… each of these terms ***is*** on the LT table:

$$= u(t) \qquad = e^{-(t/RC)}u(t)$$

$$= \left[1 - e^{-(t/RC)}\right]u(t)$$

So the zero-state response of this system is: $\left[1 - e^{-(t/RC)}\right]u(t)$



Now putting this zero-state response together with the zero-input response we found gives:

$$y(t) = y(0^-)e^{-(t/RC)}u(t) + \left[1 - e^{-(t/RC)}\right]u(t)$$

IC Part **Input Part**

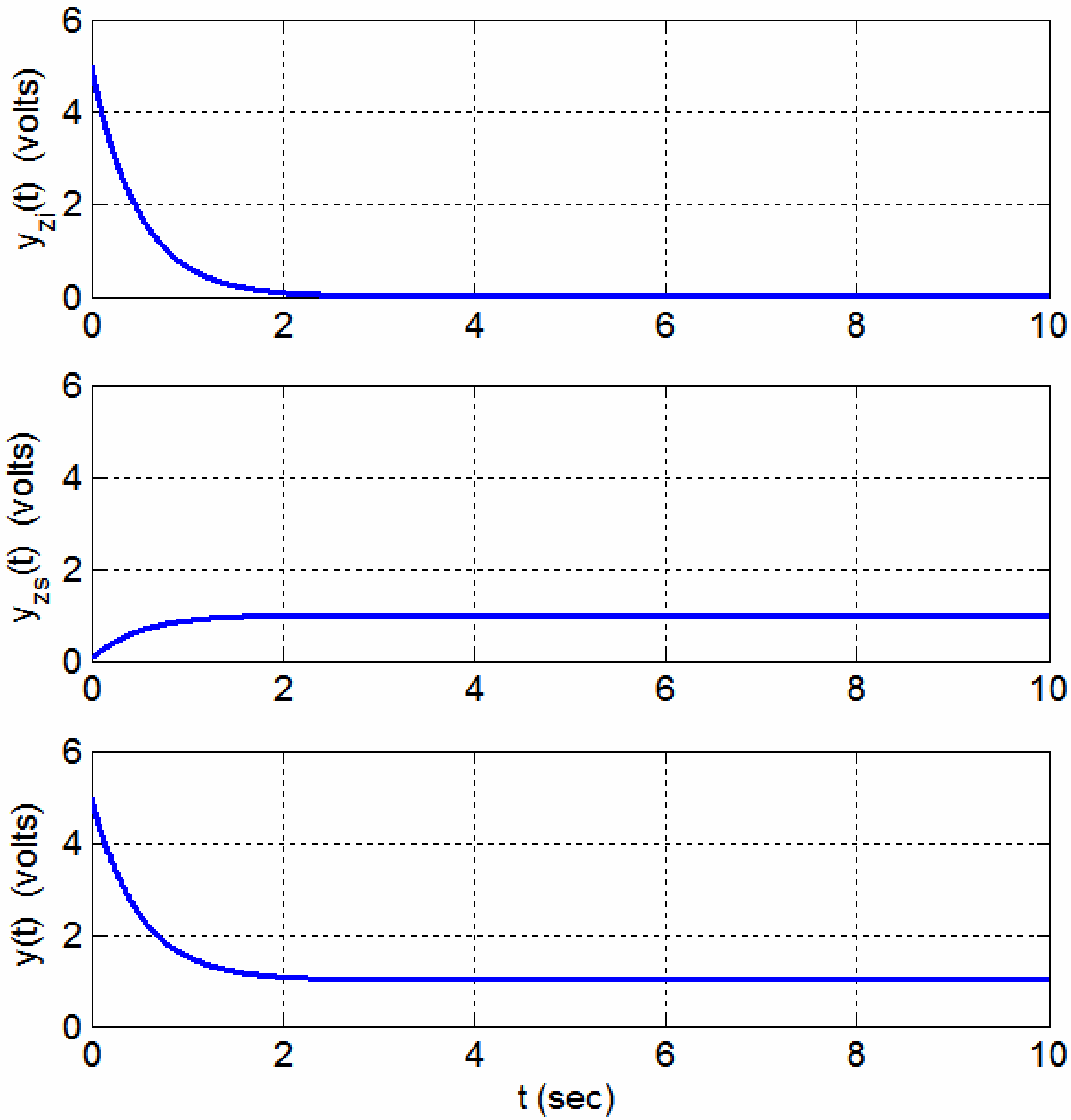
Notice that:

The IC Part "Decays Away"

but…

The Input Part "Persists"

Here is an example for ***RC* = 0.5 sec** and the initial ***V_{IC}* = 5 volts**:



Zero-Input Response

Zero-State Response

Total Response

Second-order case

Circuits with two energy-storing devices (C & L, or 2 Cs or 2 Ls) are described by a second-order Differential Equation…

$$\frac{d^2y(t)}{dt^2} + a_1\frac{dy(t)}{dt} + a_0y(t) = b_1\frac{dx(t)}{dt} + b_0x(t)$$

w/ ICs $\dot{y}(0^-)$ & $y(0^-)$

Assume Causal Input

$x(t) = 0 \quad t < 0$

$\Downarrow$

$x(0^-) = 0$

We solve the 2nd-order case using the same steps:

Take LT of Diff. Equation:

$$\left[s^2Y(s) - y(0^-)s - \dot{y}(0^-)\right] + a_1\left[sY(s) - y(0^-)\right] + a_0Y(s) = b_1sX(s) + b_0X(s)$$

From 2nd derivative property, accounting for ICs

From 1st derivative property, accounting for ICs

From 1st derivative property, causal signal

Solve for $Y(s)$: $$Y(s) = \frac{y(0^-)s + \dot{y}(0^-) + a_1 y(0^-)}{s^2 + a_1 s + a_0} + \left[\frac{b_1 s + b_0}{s^2 + a_1 s + a_0}\right] X(s)$$

Part of sol'n driven by IC
"Zero-Input Sol'n"

Note this shows up in both places… it is the <u>Characteristic Equation</u>

Part of sol'n driven by input
"Zero-State Sol'n"

<u>Note:</u> The role the <u>Characteristic Equation</u> plays here!

It just pops up in the LT method!

The same happened for a 1^{st}-order Diff. Eq…

…and it happens for all orders

Like before…

to get the solution in the time domain find the Inverse LT of Y(s)

To get a feel for this let's look at the zero-input solution for a 2nd-order system:

$$Y_{zi}(s) = \frac{y(0^-)s + \dot{y}(0^-) + a_1 y(0^-)}{s^2 + a_1 s + a_0} = \frac{y(0^-)s + \left[\dot{y}(0^-) + a_1 y(0^-)\right]}{s^2 + a_1 s + a_0}$$

which has… either a 1st-order or 0th-order polynomial in the numerator and…

… a 2nd-order polynomial in the denominator

For such scenarios there are **Two LT Pairs that are Helpful**:

$$Ae^{-\zeta\omega_n t}\sin\left[\left(\omega_n\sqrt{1-\zeta^2}\right)t\right]u(t) \quad \text{where: } A = \frac{\alpha}{\omega_n\sqrt{1-\zeta^2}} \quad \longleftrightarrow \quad \frac{\alpha}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$Ae^{-\zeta\omega_n t}\sin\left[\left(\omega_n\sqrt{1-\zeta^2}\right)t + \phi\right]u(t) \quad \longleftrightarrow \quad \beta\frac{s+\alpha}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\text{where: } A = \beta\sqrt{\frac{(\alpha-\zeta\omega_n)^2}{\omega_n^2(1-\zeta^2)} + 1}$$

$$\phi = \tan^{-1}\left(\frac{\omega_n\sqrt{1-\zeta^2}}{\alpha-\zeta\omega_n}\right)$$

For… $0 < |\zeta| < 1$

These are not in your book's table… but they are on the table on my website!

Otherwise… Factor into two terms

Note the effect of the ICs:

$$Y_{zi}(s) = \frac{y(0^-)s + \dot{y}(0^-) + a_1 y(0^-)}{s^2 + a_1 s + a_0} = \frac{y(0^-)s + \left[\dot{y}(0^-) + a_1 y(0^-)\right]}{s^2 + a_1 s + a_0}$$

If $y(0^-) = 0$:

$$Ae^{-\zeta\omega_n t} \sin\left[\left(\omega_n \sqrt{1-\zeta^2}\right)t\right] u(t) \quad \longleftrightarrow \quad \frac{\alpha}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

This form gives $y_{zi}(0) = 0$ as set by the IC

Otherwise:

$$Ae^{-\zeta\omega_n t} \sin\left[\left(\omega_n \sqrt{1-\zeta^2}\right)t + \phi\right] u(t) \quad \longleftrightarrow \quad \frac{s+\alpha}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Example of using this type of LT pair: Let $y(0^-) = 2 \quad \dot{y}(0^-) = 4$

Then

$$Y_{zi}(s) = \frac{2s + (4 + a_1 2)}{s^2 + a_1 s + a_0} = 2\left[\frac{s + (2 + a_1)}{s^2 + a_1 s + a_0}\right]$$

Pulled a 2 out from each term in Num. to get form just like in LT Pair.

Now assume that for our system we have: $a_0 = 100$ & $a_1 = 4$

Then

$$Y_{zi}(s) = 2\left[\frac{s + 6}{s^2 + 4s + 100}\right]$$

Compare to LT:

$$\beta \frac{s + \alpha}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

And identify:

$$\alpha = 6 \quad \beta = 2$$

$$\omega_n^2 = 100 \quad \Rightarrow \quad \omega_n = 10$$

$$2\zeta\omega_n = 4 \quad \Rightarrow \quad \zeta = 4/2\omega_n = 4/20 = 0.2$$

So now we use these parameters in the time-domain side of the LT pair:

$$\alpha = 6 \quad \beta = 2$$

$$\omega_n = 10$$

$$\zeta = 0.2$$

Assuming output is a voltage!

$$A = \beta\sqrt{\frac{(\alpha-\zeta\omega_n)^2}{\omega_n^2(1-\zeta^2)}+1} = 2\sqrt{\frac{(6-0.2\times10)^2}{100(1-0.2^2)}+1} = 2.16 \text{ volts}$$

$$\phi = \tan^{-1}\left(\frac{\omega_n\sqrt{1-\zeta^2}}{\alpha-\zeta\omega_n}\right) = \tan^{-1}\left(\frac{10\sqrt{1-0.2^2}}{6-0.2\times10}\right) = 1.18 \text{ rad}$$

$$Ae^{-\zeta\omega_n t}\sin\left[\left(\omega_n\sqrt{1-\zeta^2}\right)t+\phi\right]u(t)$$

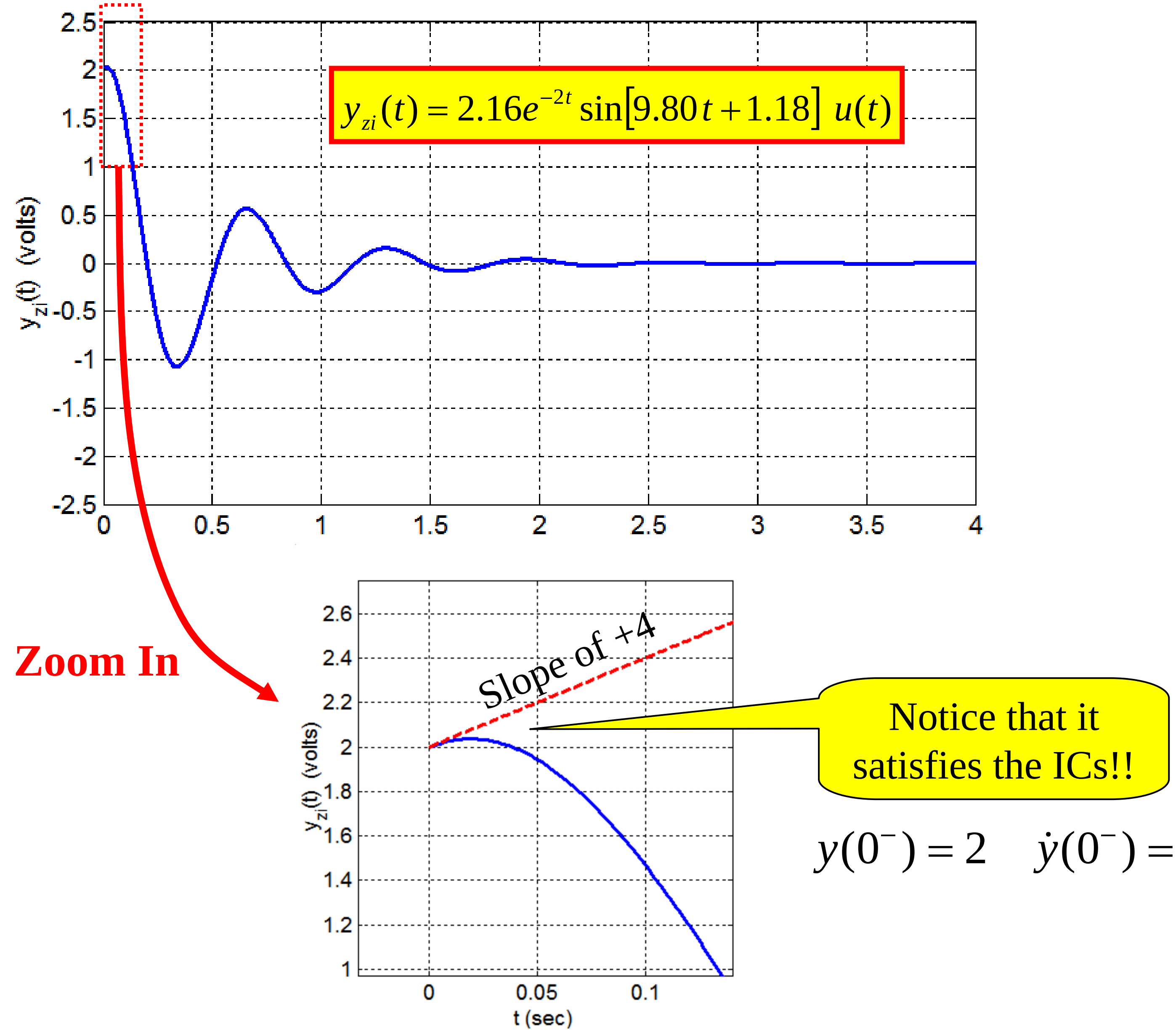
$$\text{where:} \quad A = \beta\sqrt{\frac{(\alpha-\zeta\omega_n)^2}{\omega_n^2(1-\zeta^2)}+1}$$

$$\phi = \tan^{-1}\left(\frac{\omega_n\sqrt{1-\zeta^2}}{\alpha-\zeta\omega_n}\right)$$

$$y_{zi}(t) = 2.16e^{-2t}\sin[9.80\,t+1.18]\,u(t)$$

Notice that the zero-input solution for this 2nd-order system oscillates…
1st-order systems can't oscillate…
2nd- and higher-order systems can oscillate but might not!!

Here is what this zero-input solution looks like:



$$y(0^-) = 2 \quad \dot{y}(0^-) = 4$$

N^{th}-Order Case

Diff. eq of the system

$$\frac{d^N y(t)}{dt^N} + a_{N-1}\frac{d^{N-1}y(t)}{dt^{N-1}} + \ldots + a_1\frac{dy(t)}{dt} + a_0 y(t) = b_M \frac{dx^M(t)}{dt^M} + b_1\frac{dx(t)}{dt} + b_0 x(t)$$

For $M \le N$ and $\left.\frac{d^i x(t)}{dt^i}\right|_{t=0^-} = 0 \quad i = 0,1,2,\ldots,M-1$

Taking LT and re-arranging gives:

$$Y(s) = \frac{IC(s)}{A(s)} + \frac{B(s)}{A(s)}X(s)$$

LT of the solution (i.e. the LT of the system output)

where

$$\begin{cases} A(s) = s^N + a_{N-1}s^{N-1} + \ldots + a_1 s + a_0 \\ B(s) = b_M s^M + \ldots + b_1 s + b_0 \\ IC(s) = \textit{polynomial in s that depends on the ICs} \end{cases}$$

"output-side" polynomial

"input-side" polynomial

Recall: For 2nd order case: $IC(s) = y(0^-)s + \left[\dot{y}(0^-) + a_1 y(0^-)\right]$

Consider the case where the LT of $x(t)$ is rational: $X(s) = \dfrac{N_X(s)}{D_X(s)}$

Then… $$Y(s) = \frac{IC(s)}{A(s)} + \frac{B(s)}{A(s)} X(s) = \frac{IC(s)}{A(s)} + \frac{B(s)}{A(s)} \frac{N_X(s)}{D_X(s)}$$

This can be expanded like this: $$Y(s) = \frac{IC(s)}{A(s)} + \frac{E(s)}{A(s)} + \frac{F(s)}{D_X(s)}$$

for some resulting polynomials $E(s)$ and $F(s)$

So… for a <u>system</u> with $H(s) = \dfrac{B(s)}{A(s)}$ and <u>input</u> with $X(s) = \dfrac{N_X(s)}{D_X(s)}$

and <u>initial conditions</u> you get:

$$Y(s) = \underbrace{\frac{IC(s)}{A(s)}}_{\text{Zero-Input Response}} + \underbrace{\overbrace{\frac{E(s)}{A(s)}}^{\text{Transient Response}} + \overbrace{\frac{F(s)}{D_X(s)}}^{\text{Steady-State Response}}}_{\text{Zero-State Response}}$$

Zero-Input Response

Zero-State Response

Transient Response

Steady-State Response

Decays in time domain if roots of system char. poly. *A*(*s*) have negative real parts

If all IC's are zero (zero state) $C(s) = 0$

Then:

$$Y(s) = \underbrace{\left[\frac{B(s)}{A(s)}\right]}_{\equiv H(s)} X(s)$$

Called "Transfer Function" of the system… see Sect. 6.5

Connection To sect. 6.5

Zero-State Response

$$Y(s) = \underbrace{\frac{E(s)}{A(s)}}_{\text{Transient Response}} + \underbrace{\frac{F(s)}{D_X(s)}}_{\text{Steady-State Response}}$$

<u>Summary Comments</u>:

1. From the differential equation one can easily write the $H(s)$ by inspection!

2. The denominator of $H(s)$ is the characteristic equation of the differential equation.

3.The roots of the denominator of $H(s)$ determine the <u>form</u> of the solution…

…recall partial fraction expansions

<u>BIG PICTURE:</u> The roots of the characteristic equation drive the nature of the system response… we can now see that via the LT.

We now see that there are three contributions to a system's response:

zero-input resp.

1. **The part driven by the ICs**
   a. **This will decay away if the Ch. Eq. roots have negative real parts**

zero-state resp.

2. **A part driven by the input that will decay away if the Ch. Eq. roots have negative real parts … "Transient Response"**
3. **A part driven by the input that will persist while the input persists… "Steady State Response"**