BINGHAMTON UNIVERSITY
State University of New York

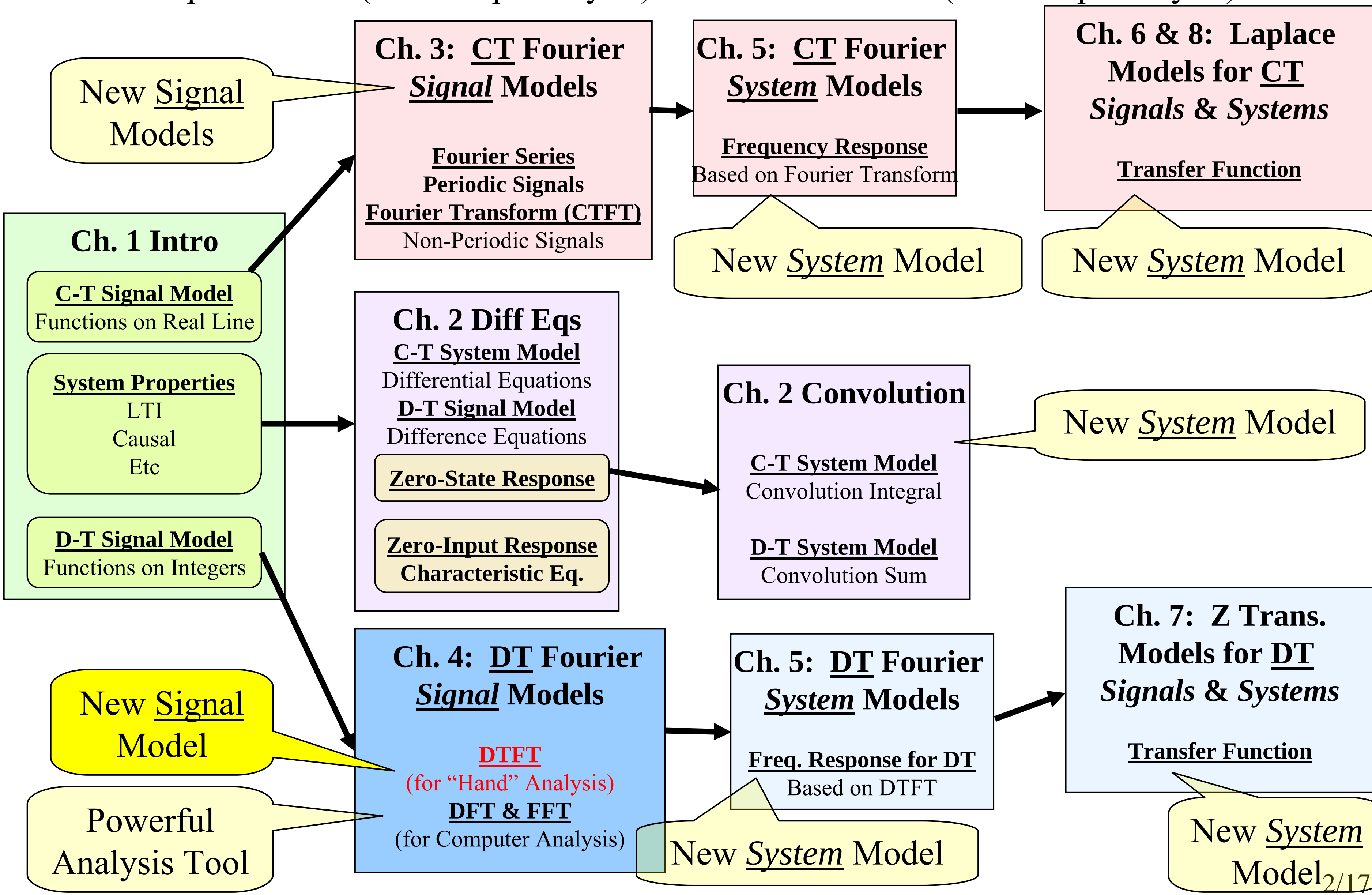
EECE 301
Signals & Systems
Prof. Mark Fowler

Note Set #23

- D-T Signals: DTFT Details
- Reading Assignment: Section 4.1 of Kamen and Heck

Course Flow Diagram

The arrows here show conceptual flow between ideas. Note the parallel structure between the pink blocks (C-T Freq. Analysis) and the blue blocks (D-T Freq. Analysis).



Sect 4.1 continued: The Details

Define the DTFT:

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\underbrace{\Omega n}}$$

In rad/sample

radians

Compare to CTFT:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\underbrace{\omega t}} dt$$

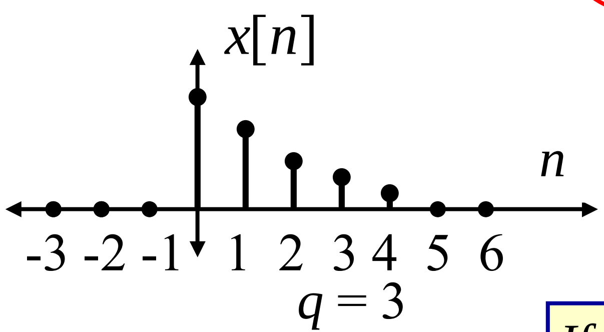
In rad/sec

radians

Very similar structure… so we should expect similar properties!!!

Example of Analytically Computing the DTFT

With your brain, not a computer



$$x[n] = \begin{cases} 0, & n < 0 \\ a^n, & 0 \le n \le q \\ 0, & n > q \end{cases}$$

$If\ |a| < 1,\quad x[n]$ decays

$If\ |a| > 1,\quad x[n]$ "explodes"

$If\ a < 0,\quad x[n]$ oscillates

Given this signal model, find the DTFT.

By definition: $X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} = \sum_{n=0}^{q} a^n e^{-j\Omega n} = \sum_{n=0}^{q} (a e^{-j\Omega})^n$

$$X(\Omega) = \frac{1 - \left(a e^{-j\Omega}\right)^{q+1}}{1 - a e^{-j\Omega}}$$

General Form for Geometric Sum:

$$\sum_{n=q_1}^{q_2} r^n = \frac{r^{q_1} - r^{q_2+1}}{1-r}$$

Characteristics of DTFT

1. Periodicity of $X(\Omega)$

$X(\Omega)$ is a periodic function of Ω with period of 2π

$$\Rightarrow X(\Omega + 2\pi) = X(\Omega)$$

Recall pictures in notes of "DTFT Intro":

$\Rightarrow |X(\Omega)|$ is periodic with period 2π

$\angle X(\Omega)$ is periodic with period 2π

Note: the CTFT does not have this property

2. $X(\Omega)$ is complex valued (in general)

$$X(\Omega) = \sum_n x[n] \underbrace{e^{-j\Omega n}}_{\text{complex}}$$

Usually think of $X(\Omega)$ in polar form:

$$X(\Omega) = \underbrace{|X(\Omega)|}_{\text{magnitude}} e^{j\overbrace{\angle X(\Omega)}^{\text{phase}}}$$

Same as CTFT

3. Symmetry

If $x[n]$ is real-valued, then:

$$|X(-\Omega)| = |X(\Omega)| \qquad \text{(even symmetry)}$$

$$\angle X(-\Omega) = -\angle X(\Omega) \qquad \text{(odd symmetry)}$$

Same as CTFT

Inverse DTFT

Q: Given $X(\Omega)$ can we find the corresponding $x[n]$?

A: Yes!!

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{jn\Omega}d\Omega$$

We can integrate instead over any interval of length 2π

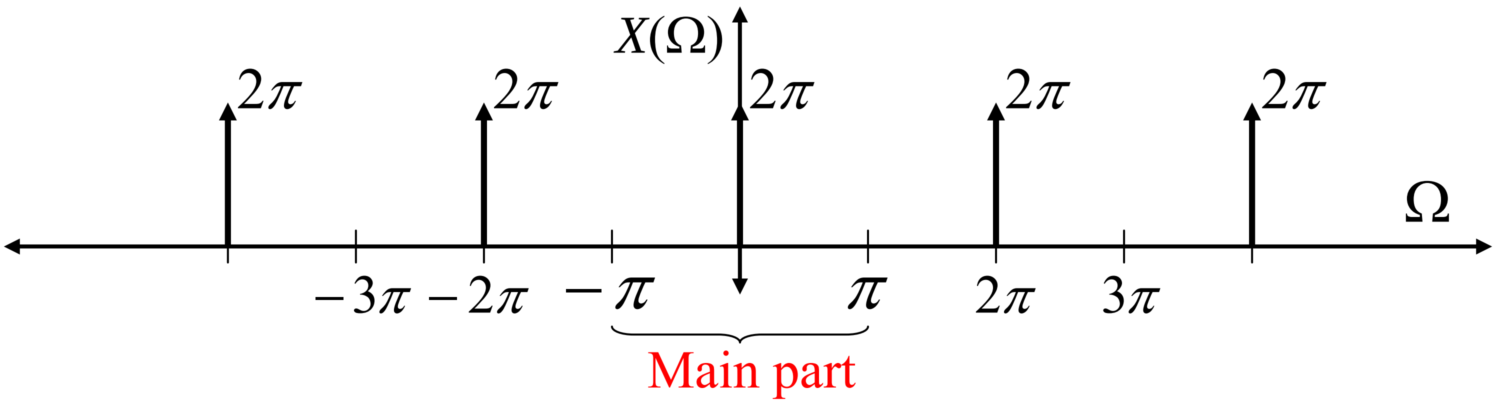
…because the DTFT is periodic with period 2π

Generalized DTFT

Periodic D-T signals have DTFT's that contain delta functions

Example: $x[n] = 1, \ \forall n \quad \leftrightarrow \quad X(\Omega) = \begin{cases} 2\pi\delta(\Omega), & -\pi < \Omega < \pi \\ periodic, & elsewhere \end{cases}$

With a period of 2π



Another way of writing this is:

$$X(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - k2\pi)$$

How do we derive the result? Work backwards!

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{jn\Omega}d\Omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\delta(\Omega)e^{jn\Omega}d\Omega$$

Sifting property

$$= e^{jn\cdot 0}$$

$$= 1$$

Transform Pairs: Like for the CTFT, there is a table of common pairs (See Web)

Be familiar with them

Compare and contrast them with the table
Of common CTFT's

Table 3.2 COMMON FOURIER TRANSFORM PAIRS

$1, \quad -\infty < t < \infty \leftrightarrow 2\pi\delta(\omega)$

$-0.5 + u(t) \leftrightarrow \dfrac{1}{j\omega}$

$u(t) \leftrightarrow \pi\delta(\omega) + \dfrac{1}{j\omega}$

$\delta(t) \leftrightarrow 1$

$\delta(t - c) \leftrightarrow e^{-j\omega c}$, c any real number

$e^{-bt}u(t) \leftrightarrow \dfrac{1}{j\omega + b}$, $b > 0$

$e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)$, ω_0 any real number

$p_\tau(t) \leftrightarrow \tau \,\text{sinc}\,\dfrac{\tau\omega}{2\pi}$

$\tau \,\text{sinc}\,\dfrac{\tau t}{2\pi} \leftrightarrow 2\pi p_\tau(\omega)$

$\left(1 - \dfrac{2|t|}{\tau}\right)p_\tau(t) \leftrightarrow \dfrac{\tau}{2}\,\text{sinc}^2\left(\dfrac{\tau\omega}{4\pi}\right)$

$\dfrac{\tau}{2}\,\text{sinc}^2\left(\dfrac{\tau t}{4\pi}\right) \leftrightarrow 2\pi\left(1 - \dfrac{2|\omega|}{\tau}\right)p_\tau(\omega)$

$\cos \omega_0 t \leftrightarrow \pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$

$\cos (\omega_0 t + \theta) \leftrightarrow \pi[e^{-j\theta}\delta(\omega + \omega_0) + e^{j\theta}\delta(\omega - \omega_0)]$

$\sin \omega_0 t \leftrightarrow j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$

$\sin (\omega_0 t + \theta) \leftrightarrow j\pi[e^{-j\theta}\delta(\omega + \omega_0) - e^{j\theta}\delta(\omega - \omega_0)]$

Table 4.1 COMMON DTFT PAIRS

$1, \text{ all } n \leftrightarrow \sum_{k=-\infty}^{\infty} 2\pi\delta(\Omega - 2\pi k)$

$\text{sgn}[n] \leftrightarrow \dfrac{2}{1 - e^{-j\Omega}}$, where $\text{sgn}[n] = \begin{cases} 1, & n = 0, 1, 2, \ldots \\ -1, & n = -1, -2, \ldots \end{cases}$

$u[n] \leftrightarrow \dfrac{1}{1 - e^{-j\Omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\Omega - 2\pi k)$

$\delta[n] \leftrightarrow 1$

$\delta[n - N] \leftrightarrow e^{-jN\Omega}$, $N = \pm 1, \pm 2, \ldots$

$a^n u[n] \leftrightarrow \dfrac{1}{1 - ae^{-j\Omega}}$, $|a| < 1$

$e^{j\Omega_0 n} \leftrightarrow \sum_{k=-\infty}^{\infty} 2\pi\delta(\Omega - \Omega_0 - 2\pi k)$

$p_q[n] \leftrightarrow \dfrac{\sin[(q + \frac{1}{2})\Omega]}{\sin(\Omega/2)}$

$\dfrac{B}{\pi}\,\text{sinc}\left(\dfrac{B}{\pi}n\right) \leftrightarrow \sum_{k=-\infty}^{\infty} p_{2B}(\Omega + 2\pi k)$

$\cos \Omega_0 n \leftrightarrow \sum_{k=-\infty}^{\infty} \pi[\delta(\Omega + \Omega_0 - 2\pi k) + \delta(\Omega - \Omega_0 - 2\pi k)]$

$\sin \Omega_0 n \leftrightarrow \sum_{k=-\infty}^{\infty} j\pi[\delta(\Omega + \Omega_0 - 2\pi k) - \delta(\Omega - \Omega_0 - 2\pi k)]$

$\cos\left(\Omega_0 n + \theta\right) \leftrightarrow \sum_{k=-\infty}^{\infty} \pi[e^{-j\theta}\delta(\Omega + \Omega_0 - 2\pi k) + e^{j\theta}\delta(\Omega - \Omega_0 - 2\pi k)]$

Careful here… the book's table doesn't have this subscript… see next slide.

DTFT of a Rectangular Pulse (Ex. 4.3)

Define: D-T pulse as $p_q[n] = \begin{cases} 1, & n = -q, \ldots, -1, 0, 1, \ldots, q \\ 0, & otherwise \end{cases}$

Book doesn't use this subscript!

Use DTFT Tables on my Website

So, by DTFT definition: $P_q(\Omega) = \sum_{n=-q}^{q} e^{-jn\Omega}$

Use "Geometric Sum" Result… see Eq. (4.5)

$$P_q(\Omega) = \frac{e^{jq\Omega} - e^{-j(q+1)\Omega}}{1 - e^{-j\Omega}} = \frac{\sin\{(q+1/2)\Omega\}}{\sin\{\Omega/2\}}$$

See book for details

Properties of the DTFT (See table on my website)

Like for the CTFT, there are many properties for the DTFT. Most are identical to those for the CTFT!!

But Note: "Summation Property" replaces Integration

There is no "Differentiation Property"

Most important ones:

- -Time shift
- -Multiplication by sinusoid… Three "flavors"
- -Convolution in the time domain
- -Parseval's Theorem

Compare and contrast these with the table of CTFT properties

Table 3.1 PROPERTIES OF THE FOURIER TRANSFORM

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \leftrightarrow aX(\omega) + bV(\omega)$
Right or left shift in time	$x(t - c) \leftrightarrow X(\omega)e^{-j\omega c}$
Time scaling	$x(at) \leftrightarrow \frac{1}{a}X\left(\frac{\omega}{a}\right) \quad a > 0$
Time reversal	$x(-t) \leftrightarrow X(-\omega) = \overline{X(\omega)}$
Multiplication by a power of t	$t^n x(t) \leftrightarrow j^n \frac{d^n}{d\omega^n}X(\omega) \quad n = 1, 2, \ldots$
Multiplication by a complex exponential	$x(t)e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0) \quad \omega_0 \text{ real}$
Multiplication by $\sin \omega_0 t$	$x(t) \sin \omega_0 t \leftrightarrow \frac{j}{2}[X(\omega + \omega_0) - X(\omega - \omega_0)]$
Multiplication by $\cos \omega_0 t$	$x(t) \cos \omega_0 t \leftrightarrow \frac{1}{2}[X(\omega + \omega_0) + X(\omega - \omega_0)]$
Differentiation in the time domain	$\frac{d^n}{dt^n}x(t) \leftrightarrow (j\omega)^n X(\omega) \quad n = 1, 2, \ldots$
Integration	$\int_{-\infty}^{t} x(\lambda)\, d\lambda \leftrightarrow \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$
Convolution in the time domain	$x(t) * v(t) \leftrightarrow X(\omega)V(\omega)$
Multiplication in the time domain	$x(t)v(t) \leftrightarrow \frac{1}{2\pi}X(\omega) * V(\omega)$
Parseval's theorem	$\int_{-\infty}^{\infty} x(t)v(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{X(\omega)}V(\omega)\, d\omega$
Special case of Parseval's theorem	$\int_{-\infty}^{\infty} x^2(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2\, d\omega$
Duality	$X(t) \leftrightarrow 2\pi x(-\omega)$

Table 4.2 PROPERTIES OF THE DTFT

Property	Transform Pair/Property
Linearity	$ax[n] + bv[n] \leftrightarrow aX(\Omega) + bV(\Omega)$
Right or left shift in time	$x[n - q] \leftrightarrow X(\Omega)e^{-jq\Omega}, \quad q \text{ any integer}$
Time reversal	$x[-n] \leftrightarrow X(-\Omega) = \overline{X(\Omega)}$
Multiplication by n	$nx[n] \leftrightarrow j\frac{d}{d\Omega}X(\Omega)$
Multiplication by a complex exponential	$x[n]e^{jn\Omega_0} \leftrightarrow X(\Omega - \Omega_0), \quad \Omega_0 \text{ real}$
Multiplication by $\sin \Omega_0 n$	$x[n] \sin \Omega_0 n \leftrightarrow \frac{j}{2}[X(\Omega + \Omega_0) - X(\Omega - \Omega_0)]$
Multiplication by $\cos \Omega_0 n$	$x[n] \cos \Omega_0 n \leftrightarrow \frac{1}{2}[X(\Omega + \Omega_0) + X(\Omega - \Omega_0)]$
Convolution in the time domain	$x[n] * v[n] \leftrightarrow X(\Omega)V(\Omega)$
Summation	$\sum_{i=0}^{n} x[i] \leftrightarrow \frac{1}{1 - e^{-j\Omega}}X(\Omega) + \sum_{n=-\infty}^{\infty} \pi X(2\pi n)\delta(\Omega - 2\pi n)$
Multiplication in the time domain	$x[n]v[n] \leftrightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega - \lambda)V(\lambda)\, d\lambda$
Parseval's theorem	$\sum_{n=-\infty}^{\infty} x[n]v[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{X(\Omega)}V(\Omega)\, d\Omega$
Special case of Parseval's theorem	$\sum_{n=-\infty}^{\infty} x^2[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lvert X(\Omega)\rvert^2\, d\Omega$
Relationship to inverse CTFT	If $x[n] \leftrightarrow X(\Omega)$ and $\gamma(t) \leftrightarrow X(\omega)p_{2\pi}(\omega)$, then $x[n] = \gamma(t)\rvert_{t=n} = \gamma(n)$

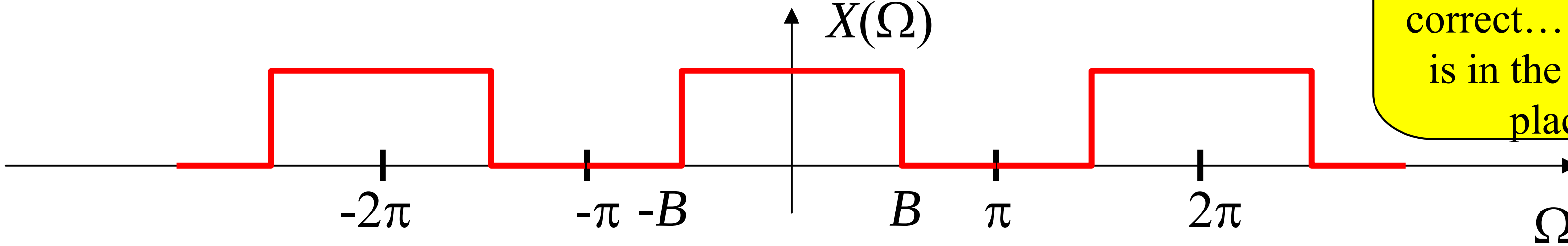
This one has no equivalent on CTFT Properties Table…
See next example

Use the Tables on my Web Site!!!

It provides a way to use a CTFT table to find DTFT pairs… here is an example

Example 4.7: Finding a DTFT pair from a CTFT pair

Book's picture is not quite correct… the "B" is in the wrong place

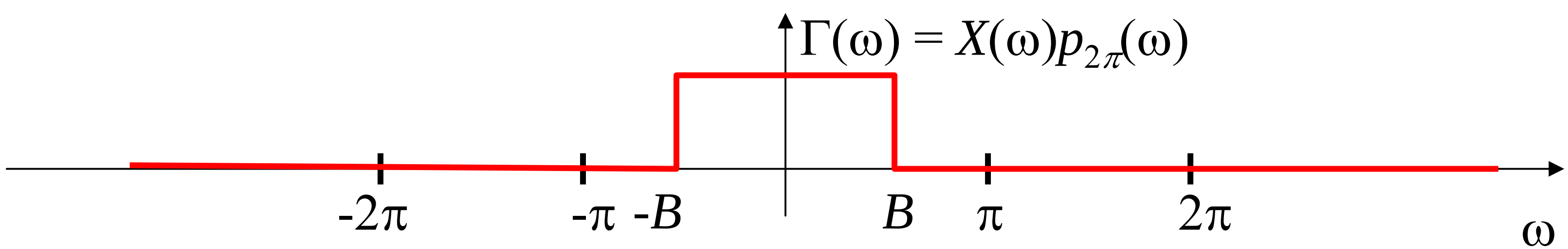


Say we are given this DTFT and want to invert it…

The four steps for using "Relationship to Inverse CTFT" property are:

1. Truncate the DTFT X(Ω) to the -π to π range and set it to zero elsewhere
2. Then treat the resulting function as a function of ω… call this Γ(ω)

$$\Gamma(\omega) = X(\omega)p_{2\pi}(\omega)$$



3. Find the inverse CTFT of $\Gamma(\omega)$ from a CTFT table, call it $\gamma(t)$

From CTFT table:

$$\gamma(t) = \frac{B}{\pi}\operatorname{sinc}\left(\frac{B}{\pi}t\right)$$

4. Get the $x[n]$ by replacing t by n in $\gamma(t)$

$$x[n] = \gamma(t)\big|_{t=n} = \frac{B}{\pi}\operatorname{sinc}\left(\frac{B}{\pi}n\right)$$

Example of DTFT of sinusoid

$$x[n] = \cos(\Omega_0 n) \quad \leftrightarrow \quad X(\Omega) = ?$$

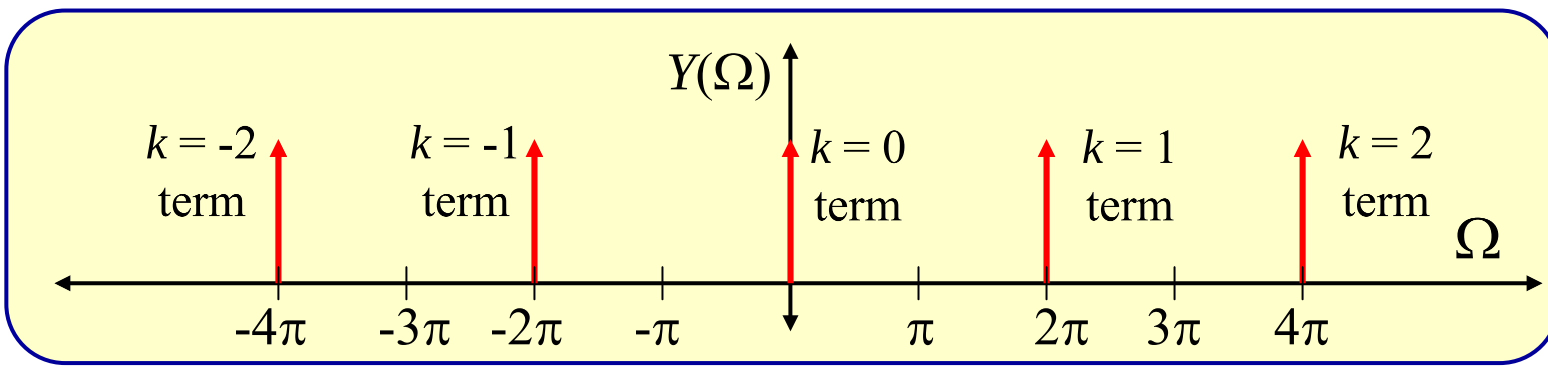
Note that: $x[n] = 1 \times \cos(\Omega_0 n)$

So… use the "mult. by sinusoid" property

$\triangleq y[n] = 1$

From DTFT Table

$$Y(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k)$$



Another way of writing this:

$$Y(\Omega) = \begin{cases} 2\pi\delta(\Omega), & -\pi < \Omega < \pi \\ 2\pi - \text{periodic elsewhere} \end{cases}$$

Recall: $x[n] = 1 \times \cos(\Omega_0 n)$ so we can use the "mult. by sinusoid" result

$$\Rightarrow X(\Omega) = \frac{1}{2}\left[Y(\Omega + \Omega_0) + Y(\Omega - \Omega_0)\right]$$

"mult. by sinusoid" property says we shift up & down by Ω_0

Using the second form for $Y(\Omega)$ gives:

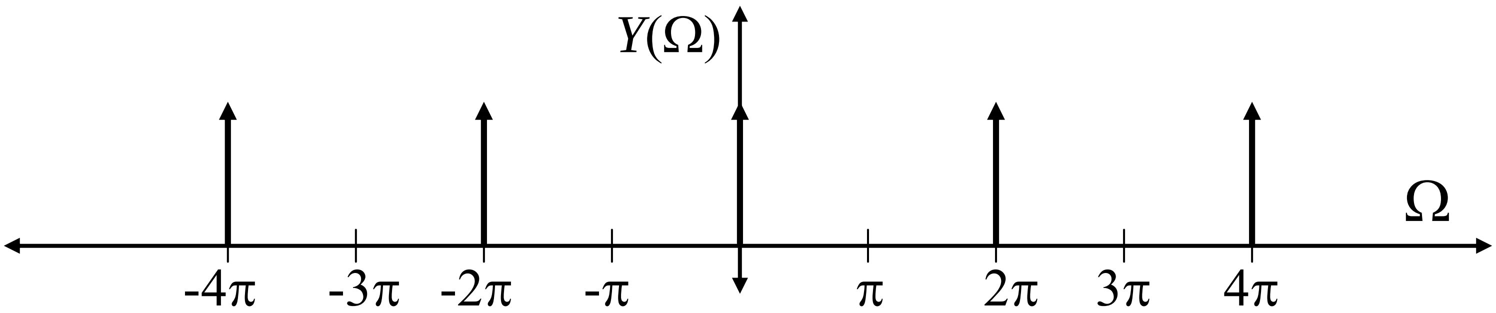
$$X(\Omega) = \begin{cases} \pi\left[\delta(\Omega + \Omega_0) + \delta(\Omega - \Omega_0)\right], & -\pi < \Omega < \pi \\ 2\pi - \text{periodic elsewhere} \end{cases}$$

Or…using the first form for $Y(\Omega)$ gives:

$$Y(\Omega) = \pi \sum_{k=-\infty}^{\infty}\left[\delta(\Omega + \Omega_0 - 2\pi k) + \delta(\Omega - \Omega_0 - 2\pi k)\right]$$

To see this graphically:

$$Y(\Omega) = \begin{cases} 2\pi\delta(\Omega), & -\pi < \Omega < \pi \\ 2\pi - \text{periodic elsewhere} \end{cases}$$



$$X(\Omega) = \begin{cases} \pi\left[\delta(\Omega+\Omega_0)+\delta(\Omega-\Omega_0)\right], & -\pi < \Omega < \pi \\ 2\pi - \textit{periodic elsewhere} \end{cases}$$