BINGHAMTON UNIVERSITY
State University of New York

EECE 301
Signals & Systems
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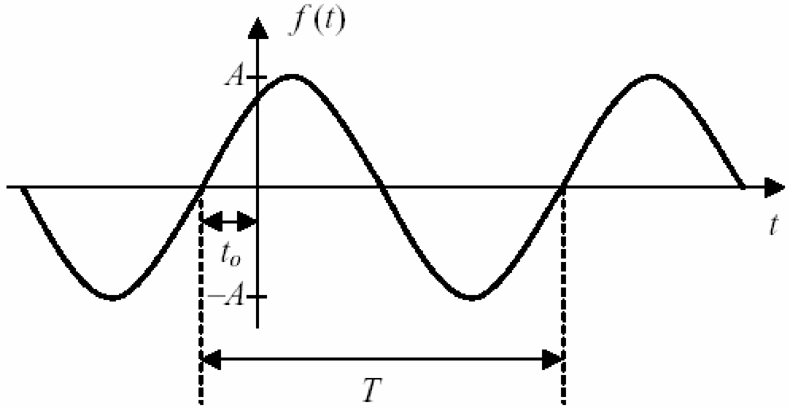
Complex Sinusoids

•Complex-Valued Sinusoidal Functions

Sinusoids

$$f(t) = A\sin(\omega t + \phi)$$



A sinusoid is completely defined by its three parameters:

- -Amplitude A (for EE's typically in volts or amps or other physical unit)
- -Frequency ω in radians per second
- -Phase shift ϕ in radians

T is the period of the sinusoid and is related to the frequency

Frequency can be expressed in two common units:

- -Cyclic frequency: $f = 1/T$ in Hz (1 Hz = 1 cycle/second)
- -Radian Frequency: $\omega = 2\pi/T$ (in radians/second)

From this we can see that these two frequency units have a simple conversion factor relationship (like all other unit conversions – e.g. feet and meters):

$$\omega = 2\pi f$$

Phase shift (often just shortened to phase) shows up explicitly in the equation but shows up in the plot as a time shift (because the plot is a function of time).

Q: What is the relationship between the plot-observed time shift and the equation-specified phase shift?

A: We can write the time shift of a function by replacing t by $t + t_o$ (more on this later, but you should be able to verify that this is true!) Then we get:

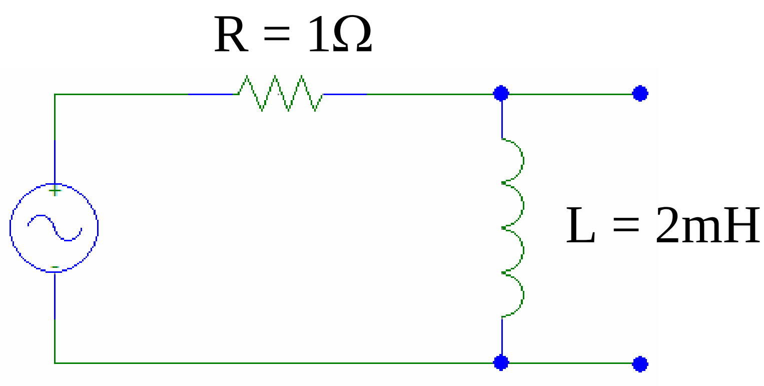
$$f(t + t_o) = A\sin(\omega(t + t_o)) = A\sin(\omega t + \underbrace{\omega t_o}_{= \phi})$$

So we get that: $\phi = \omega t_o$ (unit-wise this makes sense!!!)

In circuits you used "phasors" (we'll call them "static" phasors here)… the point of using them is to make it EASY to analyze circuits that are driven by a single sinusoid. Here is an example to refresh your memory!!

Find output voltage of the following circuit:

$x(t) = 5\cos\left(1000t + \pi/4\right)$



$y(t) = ?$

Use phasor and impedance ideas:

Impedance of Inductor: $Z_L = j\omega L = j2$

Phasor of Input: $\hat{x} = 5e^{j\pi/4}$

Use voltage divider to find output:

$$\hat{y} = \hat{x}\left[\frac{j2}{1+j2}\right] = 5e^{j\pi/4}\left[0.89e^{j0.46}\right]$$

Output phasor:

$$\hat{y} = 4.45e^{j1.25}$$

Output signal: $y(t) = 4.45\cos(1000t + 1.25)$

Note that in using "static" phasors there was no need to "carry around the frequency" … it gets suppressed in the static phasor

BUT… if you have multiple driving sinusoids (each at its own unique frequency) then you'll need to keep that frequency in the phasor representation… that leads to:

Rotating Phasors

Keeping the frequency "part"

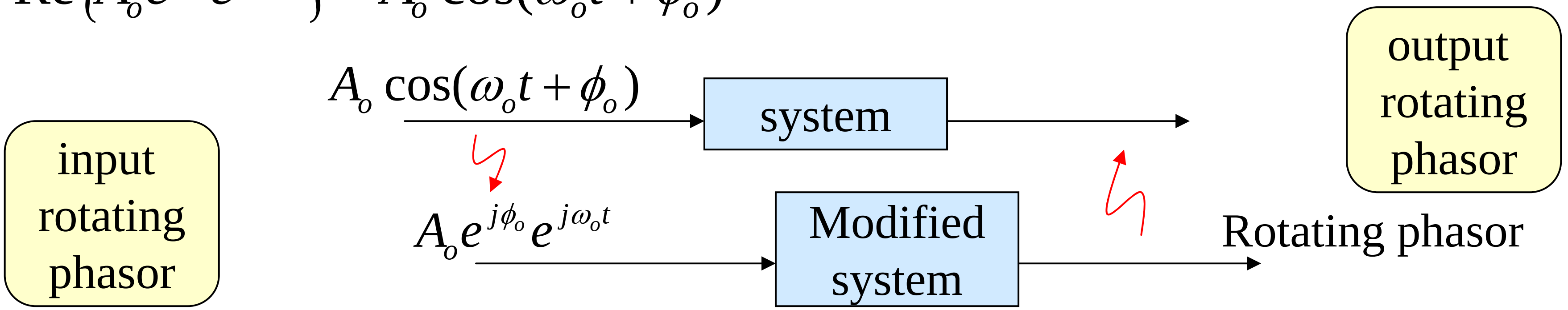
$$A_o \cos(\omega_o t + \phi_o) \rightarrow \boxed{A_o e^{j(\omega_o t + \phi_o)}}$$

rotating phasor

$$= \underbrace{A_o e^{j\phi_o}}_{\text{"static" phasor part}} e^{j\omega_o t}$$

$$e^{j\omega_o t} = \cos(\omega_o t) + j\sin(\omega_o t)$$

$$\text{Re}\left\{A_o e^{j\phi} e^{j\omega_o t}\right\} = A_o \cos(\omega_o t + \phi_o)$$



If $\tilde{x}(t) = Ae^{+j[\omega_o t + \phi_o]}$

$\left(x(t) = A\cos(\omega_o t + \phi_o)\right)$

What is: $\tilde{x}^*(t)$

$$= Ae^{-j[\omega_o t + \phi_o]}$$

$$= Ae^{-j\phi_o}e^{-j\omega_o t}$$

$$\frac{\tilde{x}(t) + \tilde{x}^*(t)}{2} = \not{2}\,\mathrm{Re}\{\tilde{x}(t)\}$$

$$= \not{2}\,A\cos(\omega_o t + \phi_o)$$

Because rotating phasors take the value of a complex number at each Instant of time they must follow all the rules of complex numbers…
Especially: EULER’S EQUATIONS!!

Rotating Phasors… Complex Sinusoids

Euler's Equations

$$e^{j(\omega t+\theta)} = \cos(\omega t+\theta) + j\sin(\omega t+\theta) \qquad \text{(A)}$$

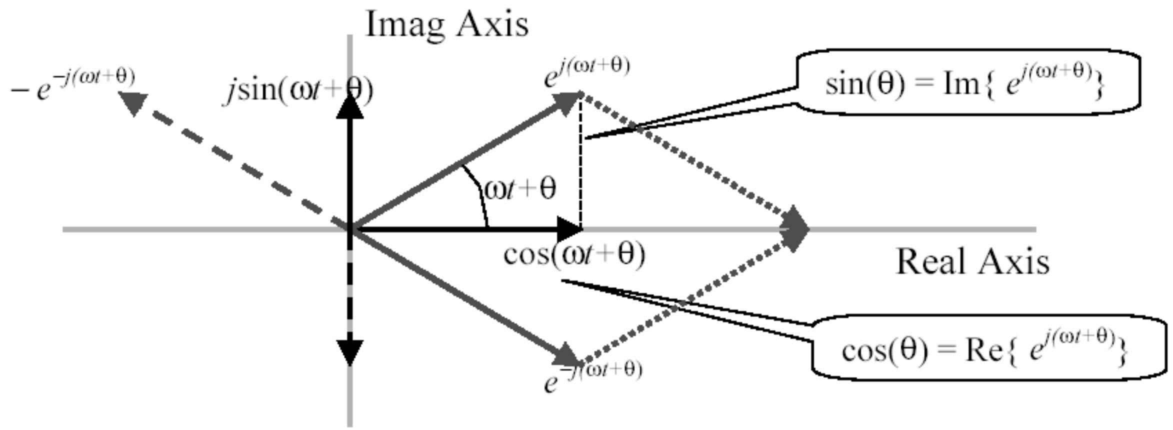
$$e^{-j(\omega t+\theta)} = \cos(\omega t+\theta) - j\sin(\omega t+\theta) \qquad \text{(B)}$$

$$\cos(\omega t+\theta) = \frac{e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}}{2} \qquad \text{(C)}$$

$$\sin(\omega t+\theta) = \frac{e^{j(\omega t+\theta)} - e^{-j(\omega t+\theta)}}{2j} \qquad \text{(D)}$$

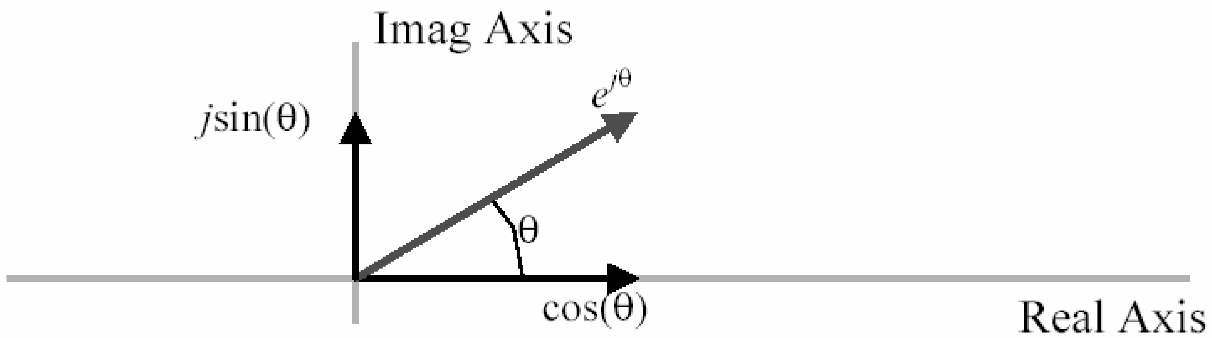
Note: Eq. C = (Eq. A+ Eq. B)/2 D = (A – B)/2

A = C + jD B = C – jD

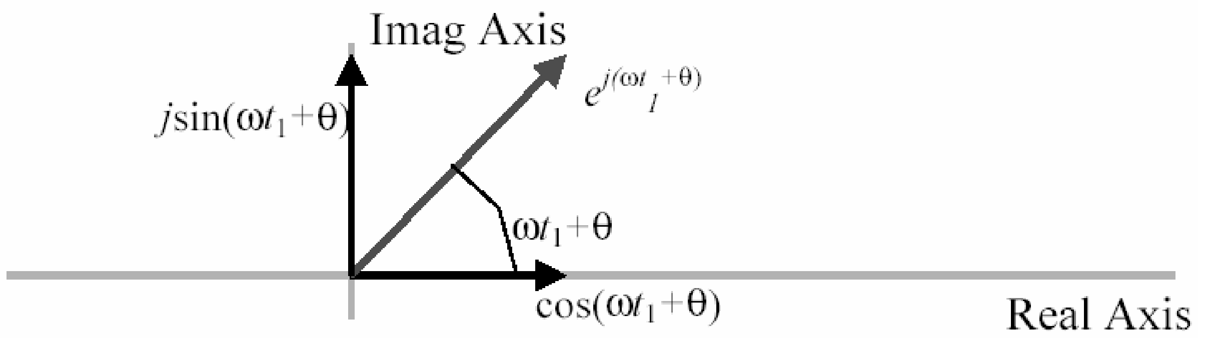


$t = 0$

Viewing rotating phasor on the complex plane



$t = t_1 > 0$



$t = t_2 > t_1$