BINGHAMTON
UNIVERSITY
State University of New York

EECE 301
Signals & Systems
Prof. Mark Fowler

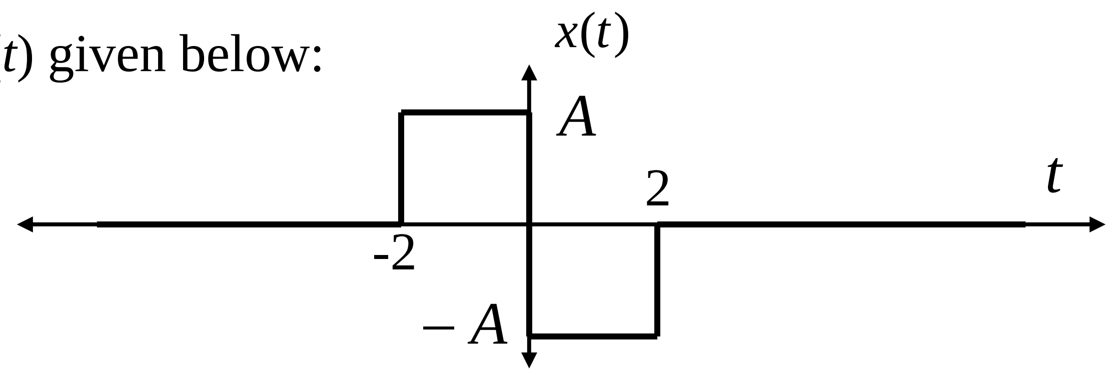
Discussion #6

- Fourier Transform Examples

FT Examples

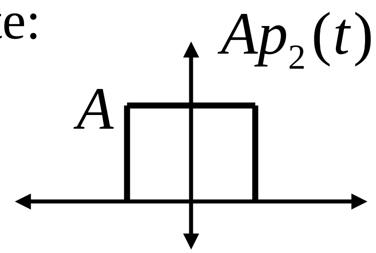
Example:

Find FT of $x(t)$ given below:



Solution:

Note:



$$\text{So: } x(t) = Ap_2(t+1) - Ap_2(t-1)$$

$$\text{Use Linearity: } \mathfrak{F}\{x(t)\} = A\mathfrak{F}\{p_2(t+1)\} - A\mathfrak{F}\{p_2(t-1)\}$$

$$\text{Use Time Shift: } \mathfrak{F}\{p_2(t+1)\} = P_2(\omega)e^{j\omega}$$

$$\mathfrak{F}\{p_2(t-1)\} = P_2(\omega)e^{-j\omega}$$

$$\text{So...} \quad X(\omega) = AP_2(\omega)\left[e^{j\omega} - e^{-j\omega}\right]$$

$$= AP_2(\omega)2j\left[\frac{e^{j\omega} - e^{-j\omega}}{2j}\right]$$

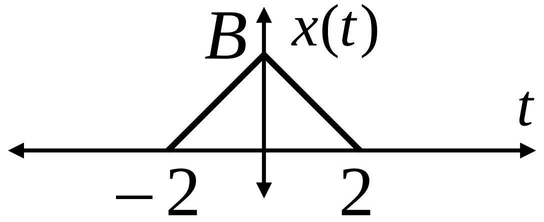
$$= A2j\sin(\omega)P_2(\omega)$$

$$\text{From Table: } P_2(\omega) = 2\text{sinc}\left(\frac{\omega}{\pi}\right)$$

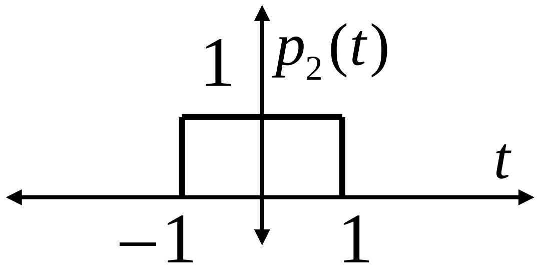
$$X(\omega) = A4j\sin(\omega)\text{sinc}\left(\frac{\omega}{\pi}\right)$$

<u>Example</u>

Find FT of $x(t)$



<u>Solution #1</u>



<u>Note:</u> $x(t) = \left(p_2(t) * p_2(t)\right)\dfrac{B}{2}$ Verify it!!

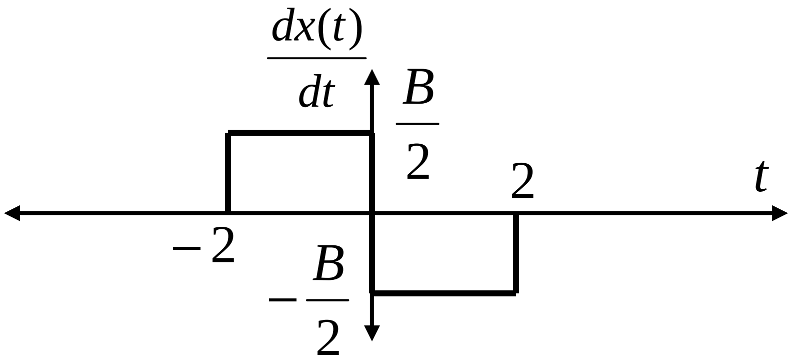
Using Convolution Property: $X(\omega) = \dfrac{B}{2} P_2^{\,2}(\omega)$

From Table: $P_2(\omega) = 2\operatorname{sinc}\left(\dfrac{2\omega}{2\pi}\right) = 2\operatorname{sinc}\left(\dfrac{\omega}{\pi}\right)$

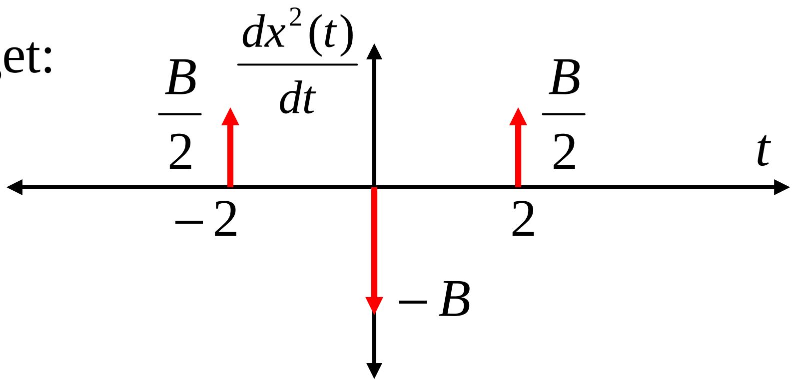
So... $X(\omega) = 2B\operatorname{sinc}^2\left(\dfrac{\omega}{\pi}\right)$

<u>Solution #2</u>

Take $\dfrac{dx(t)}{dt}$ to get



Take another derivative to get:



Now by Linearity and $\mathfrak{F}\{\delta(t)\}$ & time shift:

$$\mathfrak{F}\left\{\frac{dx^2(t)}{dt}\right\} = \frac{B}{2}\mathfrak{F}\{\delta(t+2)\} - B\mathfrak{F}\{\delta(t)\} + \frac{B}{2}\mathfrak{F}\{\delta(t-2)\}$$

$$= B\left[\frac{1}{2}e^{j\omega 2} - 1 + \frac{1}{2}e^{-j\omega 2}\right] = B\big(\cos(2\omega) - 1\big) = -2B\sin^2(\omega)$$

Euler! ⇒ cos(ω2)

Now by derivative property:

$$\mathscr{F}\left\{\frac{d^2x(t)}{dt}\right\} = \underbrace{(j\omega)^2}_{=-\omega^2} X(\omega) \Rightarrow X(\omega) = \frac{-1}{\omega^2}\mathscr{F}\left\{\frac{d^2x(t)}{dt}\right\}$$

$$\Rightarrow X(\omega) = \frac{-1}{\omega^2}\left[-2B\sin^2(\omega)\right]$$

$$= 2B\left[\frac{\sin(\omega)}{\omega}\right]^2 = 2B\left[\frac{\sin\left(\pi\frac{\omega}{\pi}\right)}{\pi\cdot\frac{\omega}{\pi}}\right]^2$$

$$\boxed{X(\omega) = 2B\text{sinc}^2\left(\frac{\omega}{\pi}\right)}$$

Same Result!!

Semi-real-world example

-We want to wirelessly send binary data

-Suppose FCC has allocated the band centered at 50MHz with bandwidth of 20kHz

$= f_o$

$\omega_o = 2\pi f_o \Rightarrow$ Freq. Response $= [49.99 \quad 50.01]$MHz

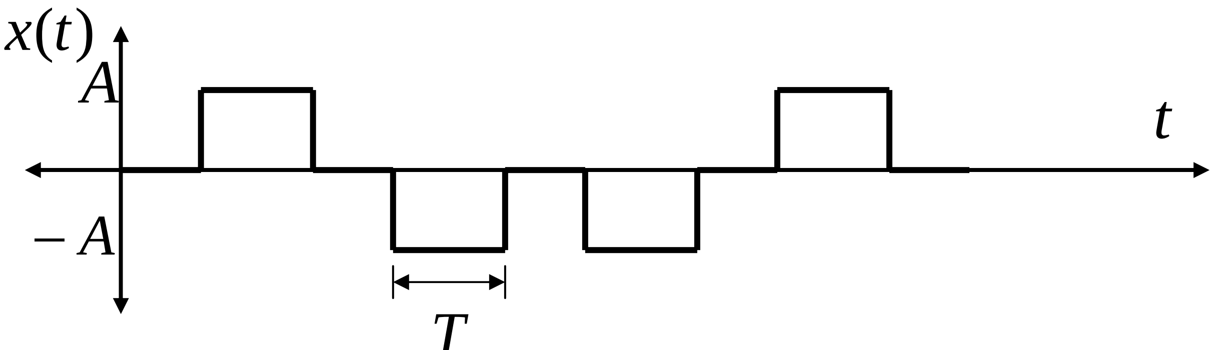
$= f_1$ $\quad$ $= f_2$

$\omega_1 = 2\pi f_1$

$\omega_2 = 2\pi f_2$

Suppose…"FCC Rule": No more than 2% of energy outside band

-Suppose you wish to use a positive, rect. pulse to send a "1" and a negative, rect. pulse to send a "0"



Clearly, the smaller you make T the faster you can send data: $\quad \downarrow T = \uparrow$Data Rate

Q: How fast can you send data while adhering to "FCC Rule"?

Rough Solution

Build $x(t)$ from shifted rect. pulses: $$x(t) = \sum_{n=0}^{N} A_n p_T(t - nT)\cos(\omega_o t)$$

$A_n = \pm A$ for 1 on 0

By linearity of FT, modulation prop., & time shift property:

$$X(\omega) = \sum_{n=0}^{N} A_n \left[\frac{P_t(\omega + \omega_o) + P_t(\omega - \omega_o)}{2} \right] e^{-jnT\omega}$$

Modulation prop. Delay prop.

To get a rough ideal of the answer:

$\Rightarrow$ Look at energy of one pulse $p_T(t)\cos(\omega_o t)$

$\Rightarrow$ We seek to ensure no more than 2% of energy of $p_t(t)\cos(\omega_o t)$ itself lies outside the band. (In EECE 377/477) you'll learn more precise ways to do this)

Total energy of pulse: $E = \int_{-T/2}^{T/2} (\pm A)^2 \cos^2(\omega_o t) dt$

$$let \qquad x = \omega_o t \quad \Rightarrow t = \frac{1}{\omega_o} x \qquad dt = \frac{1}{\omega_o} dx$$

$$t = \pm T/2 \quad \Rightarrow x = \pm \frac{\omega_o T}{2}$$

$$\Rightarrow E = \frac{A^2}{\omega_o} \int_{-\frac{\omega_o T}{2}}^{\frac{\omega_o T}{2}} \cos^2(x) dx = \frac{A^2}{\omega_o} \left[\frac{1}{4} \sin(2x) + \frac{1}{2} x \right]_{-\frac{\omega_o T}{2}}^{\frac{\omega_o T}{2}}$$

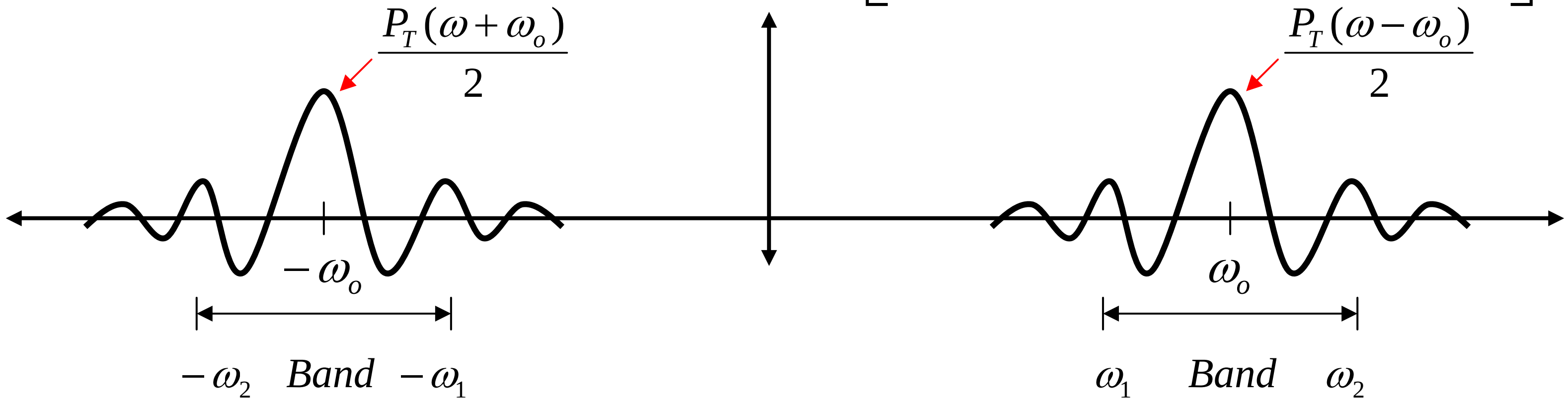
$$E = \frac{2A^2}{\omega_o} \left[\frac{1}{4} \sin\left(\frac{\omega_o T}{2} \right) + \frac{1}{2} \frac{\omega_o T}{2} \right]$$

← So 98% of this must be in band: $0.98E$

Okay…set that result aside. Look at FT of one pulse (Note: since delay does not affect the magnitude of the FT it does not affect energy)

$$P(\omega) \triangleq \mathfrak{F}\{A p_T(t) \cos(\omega_o t)\} = A \left[\frac{P_T(\omega + \omega_o) + P_T(\omega - \omega_o)}{2} \right]$$

$$P(\omega) \equiv \mathfrak{F}\{Ap_T(t)\cos(\omega_o t)\} = A\left[\frac{P_T(\omega+\omega_o) + P_T(\omega-\omega_o)}{2}\right]$$



$$|P(\omega)|^2 = \frac{A^2}{4}\left[P_T(\omega+\omega_o) + P_T(\omega-\omega_o)\right]^2$$

(Ignore because it is small)

$$= \frac{A^2}{4}P_T{}^2(\omega+\omega_o) + \frac{A^2}{4}P_T{}^2(\omega-\omega_o) + \frac{1}{8}P_T(\omega+\omega_o)P_T(\omega-\omega_o)$$

$(P_T(\omega-\omega_o) \approx 0$ where $P_T(\omega+\omega_o) \neq 0$ and vice versa)

<u>Now:</u> $$E_{band} = \int_{+\omega_1}^{+\omega_2} |P(\omega)|^2 d\omega + \int_{-\omega_1}^{-\omega_2} |P(\omega)|^2 d\omega$$

Parseval!!

But by even symmetry of magnitude of FT these two terms are the same!

$$\Rightarrow E_{Band} = 2\int_{\omega_1}^{\omega_2} \left|P(\omega)\right|^2 d\omega$$

$$\approx \frac{2A^2}{4}\int_{\omega_1}^{\omega_2}\Big[\underbrace{P_T{}^2(\omega+\omega_o)}_{\approx 0 \text{ over } \omega\in[\omega_1,\omega_2]} + P_T{}^2(\omega-\omega_o)\Big]d\omega$$

let $\zeta = \omega_1 - \omega_0 \Rightarrow d\zeta = d\omega$

$$\omega = \omega_1 \to \zeta = \omega_1 - \omega_0 < 0 \text{ because } \omega_1 < \omega_0$$

$$\omega = \omega_2 \to \zeta = \omega_2 - \omega_0 < 0 \text{ because } \omega_2 > \omega_0$$

$$= \frac{A^2}{2}\int_{\omega_1-\omega_0 \triangleq -\Delta\omega}^{\omega_2-\omega_0 \triangleq \Delta\omega} P_T{}^2(\zeta)d\zeta$$

$$= \frac{A^2}{2}\int_{-\Delta\omega}^{\Delta\omega} T^2 \text{sinc}^2\left(\frac{T\omega}{2\pi}\right)d\omega$$

Just re-write again in ω… who cares…

ζ… ω… both just dummy variables!

Now…

$$\mathrm{sinc}\left(\frac{T\omega}{2\pi}\right) = \frac{\sin\left(\frac{T\omega}{2}\right)}{\frac{T\omega}{2}}$$

$$E_{Band} = \frac{A^2T^2}{2}\int_{-\Delta\omega}^{\Delta\omega} \frac{\sin^2\left(\frac{T\omega}{2}\right)}{\left(\frac{T\omega}{2}\right)^2} d\omega$$

Nasty Integral!!

Then… resort to "numerical integration"

Note: We want $E_{\text{band}}/E = 0.98$

$\Rightarrow A^2$ in top and bottom cancels!

$\Rightarrow$ Ignore A ($A = 1$)

1. Compute $E = \frac{2}{\omega_o}\left[\frac{1}{4}\sin\left(\frac{\omega_o T}{2}\right) + \frac{1}{2}\frac{\omega_o T}{2}\right]$

for $\omega_o = 2\pi \times 50 \times 10^6$ as a function of T

2. Compute E_{band} (using numerical integration - use Matlab)

3. Plot $\frac{E_{band}}{E}$ vs. T