BINGHAMTON UNIVERSITY
State University of New York

EECE 301
Signals & Systems
Prof. Mark Fowler

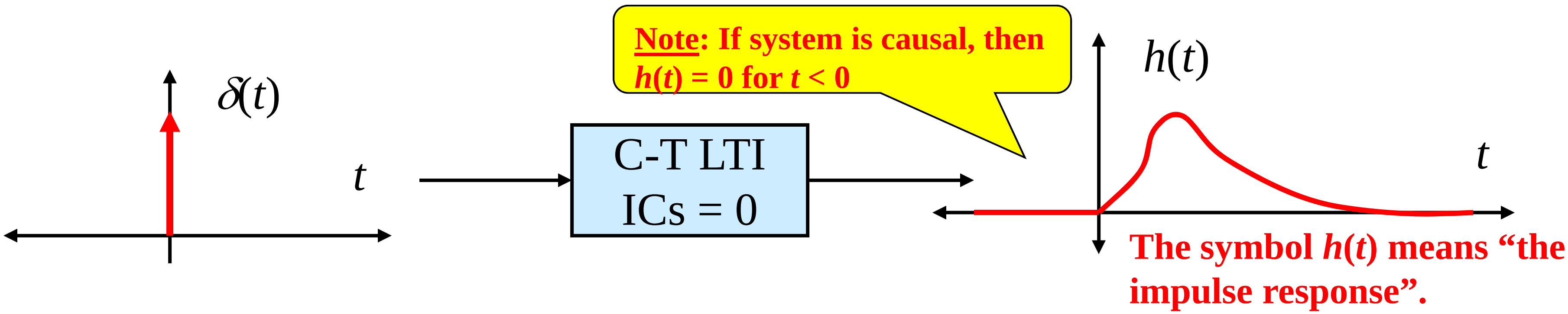
Note Set #38

- C-T Convolution: The Tool for Finding the Zero-State Response

Recall: Impulse Response

Earlier we introduced the concept of impulse response…

…what comes out of a system when the input is an impulse (delta function)



Noting that the LT of $\delta(t) = 1$ and using the properties of the transfer function and the Z transform we said that

$$h(t) = \mathscr{L}^{-1}\{H(s)\mathscr{L}\{\delta(t)\}\}$$

$$h(t) = \mathscr{L}^{-1}\{H(s)\}$$

$$h(t) = \mathscr{F}^{-1}\{H(\omega)\}$$

So…once we have either $H(s)$ or $H(\omega)$ we can get the impulse response $h(t)$

Since $H(s)$ & $H(\omega)$ describe the system so must the impulse response $h(t)$

How???

Convolution Property and System Output

Let $x(t)$ be a signal with CTFT $X(\omega)$ and LT of $X(s)$

$$x(t) \leftrightarrow X(\omega)$$
$$x(t) \leftrightarrow X(s)$$

Consider a system w/ freq resp $H(\omega)$ & trans func $H(s)$

$$h(t) \leftrightarrow H(\omega)$$
$$h(t) \leftrightarrow H(s)$$

We've spent much time using these tools to analyze system outputs this way:

$$Y(\omega) = H(\omega)X(\omega) \quad \leftrightarrow \quad y(t) = \mathscr{F}^{-1}\{H(\omega)X(\omega)\}$$

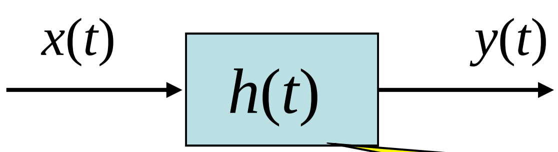
$$Y(s) = H(s)X(s) \quad \leftrightarrow \quad y[n] = \mathscr{L}^{-1}\{H(s)X(s)\}$$

The convolution property of the CTFT and LT gives an alternate way to find *y*(*t*):

$$\mathscr{F}^{-1}\{X(\omega)H(\omega)\} = x(t) * h(t)$$

$$\mathscr{L}^{-1}\{X(s)H(s)\} = x(t) * h(t)$$

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$



LTI System with impulse response *h*(*t*)

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

"Convolving" input *x*(*t*) with the impulse response *h*(*t*) gives the output *y*(*t*)!

<u>Convolution for Causal System & with Causal Input</u>

An arbitrary LTI system's output can be found using the general convolution form:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

General LTI System

If the system is causal then $h(t) = 0$ for $t < 0$…. Thus $h(t - \tau) = 0$ for $t > \tau$ … so:

$$y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau$$

Causal LTI System

If the input is causal then $x(t) = 0$ for $t < 0$…. so:

$$y(t) = \int_{0}^{\infty} x(\tau)h(t-\tau)d\tau$$

Causal Input & General LTI System

If the system & signal are both causal then

$$y(t) = \int_{0}^{t} x(\tau)h(t-\tau)d\tau$$

Causal Input & Causal LTI System

Convolution Properties

First Three Identical to DT Case!!

1. Commutativity $x(t)*h(t) = h(t)*x(t)$

2. Associativity $[x(t)*h_1(t)]*h_2(t) = x(t)*[h_1(t)*h_2(t)]$

Associativity together with commutativity says we **can interchange the order of two cascaded systems**.

3. Distributivity $x(t)*[h_1(t)+h_2(t)] = x(t)*h_1(t) + x(t)*h_2(t)$

4. Derivative Property:

$$\frac{d}{dt}[x(t)*v(t)] = \dot{x}(t)*v(t)$$
$$= x(t)*\dot{v}(t)$$

derivative

5. Integration Property Let $y(t) = x(t)*h(t)$, then

$$\int_{-\infty}^{t} y(\lambda)d\lambda = \left[\int_{-\infty}^{t} x(\lambda)d\lambda\right]*h(t) = x(t)*\left[\int_{-\infty}^{t} h(\lambda)d\lambda\right]$$

Steps for Graphical Convolution $x(t)*h(t)$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Note: I use τ for what the book uses λ... It is not a big deal as they are just dummy variables!!!

1. **Re-Write the signals as functions of τ:** $x(\tau)$ and $h(\tau)$

2. **Flip** just one of the signals around $t = 0$ to get either $x(-\tau)$ or $h(-\tau)$
   a. It is usually best to flip the signal with shorter duration
   b. For notational purposes here: we'll flip $h(\tau)$ to get $h(-\tau)$

3. **Find Edges** of the flipped signal
   a. Find the left-hand-edge τ-value of $h(-\tau)$: **call it $\tau_{L,0}$**
   b. Find the right-hand-edge τ-value of $h(-\tau)$: **call it $\tau_{R,0}$**

4. **Shift** $h(-\tau)$ by an arbitrary value of t to get $h(t - \tau)$ and **get its edges**
   a. Find the left-hand-edge τ-value of $h(t - \tau)$ as a function of t: **call it $\tau_{L,t}$**
      - **Important:** It will *always* be… $\tau_{L,t} = t + \tau_{L,0}$

   b. Find the right-hand-edge τ-value of $h(t - \tau)$ as a function of t: **call it $\tau_{R,t}$**
      - **Important:** It will *always* be… $\tau_{R,t} = t + \tau_{R,0}$

Note: If the signal you flipped is NOT finite duration, one or both of $\tau_{L,t}$ and $\tau_{R,t}$ will be infinite ($\tau_{L,t} = -\infty$ and/or $\tau_{R,t} = \infty$)

Steps Continued

5. <u>Find Regions of τ-Overlap</u>

a. What you are trying to do here is find intervals of t over which the product $x(\tau)\, h(t - \tau)$ has a single mathematical form in terms of τ

b. In each region find: Interval of t that makes the identified overlap happen

c. Working examples is the best way to learn how this is done

<u>Tips</u>: Regions should be contiguous with no gaps!!!
Don't worry about $<$ vs. $\leq$ etc.

6. For Each Region: **<u>Form the Product $x(\tau)\, h(t - \tau)$ and Integrate</u>**

a. <u>Form product</u> $x(\tau)\, h(t - \tau)$

b. <u>Find the Limits of Integration</u> by finding the interval of τ over which the product is nonzero

   i. Found by seeing where the edges of $x(\tau)$ and $h(t - \tau)$ lie

   ii. Recall that the edges of $h(t - \tau)$ are $\tau_{L,t}$ and $\tau_{R,t}$, which often depend on the value of t

      • So… the limits of integration <u>may</u> depend on t

c. <u>Integrate the product</u> $x(\tau)\, h(t - \tau)$ over the limits found in 6b

   i. The result is generally a function of t, but is only valid for the interval of t found for the current region

   ii. Think of the result as a "time-section" of the output $y(t)$

Steps Continued

7. <u>**"Assemble" the output**</u> from the output time-sections for all the regions

a. Note: you do NOT add the sections together

b. You define the output "piecewise"

c. Finally, if possible, look for a way to write the output in a simpler form

Example: Graphically Convolve Two Signals

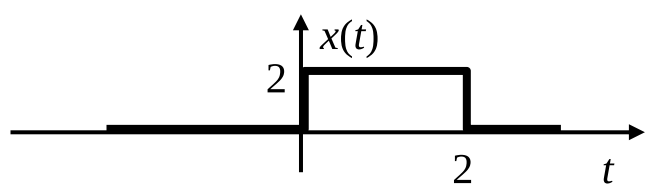
$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

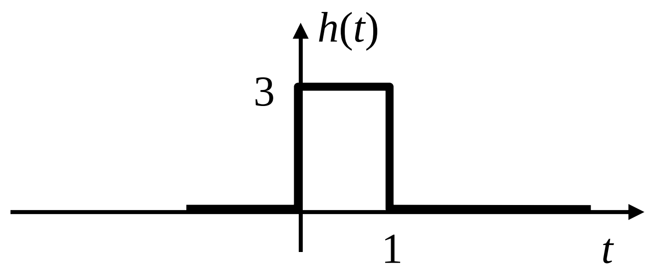
$$= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

By "Properties of Convolution"… these two forms are Equal

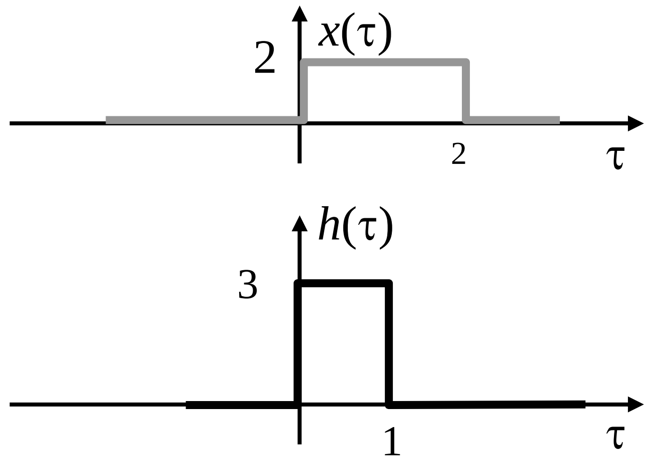
<u>This</u> is why we can flip <u>either</u> signal

Convolve these two signals:

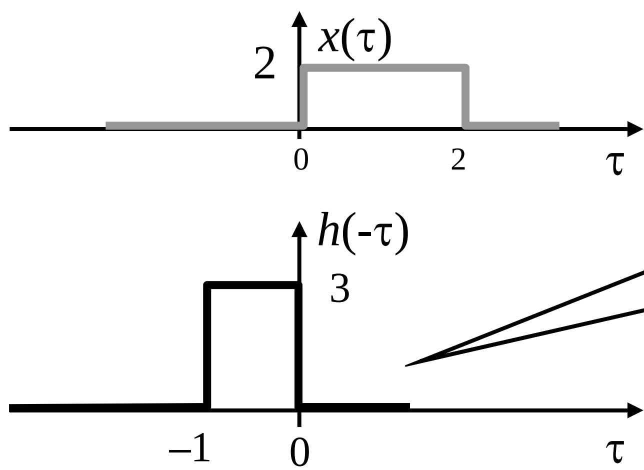




Step #1: Write as Function of τ

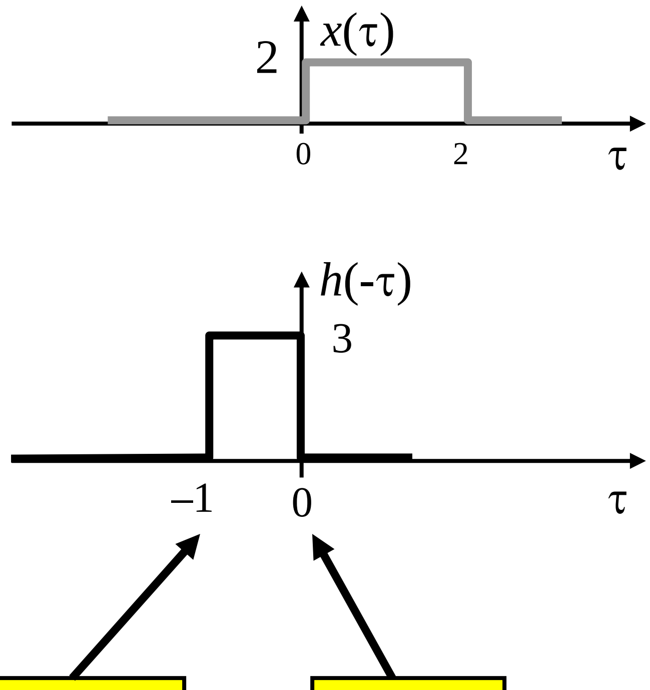


Step #2: Flip *h*(τ) to get *h*(-τ)



Usually Easier to Flip the Shorter Signal

Step #3: Find Edges of Flipped Signal



$\tau_{L,0} = -1$

$\tau_{R,0} = 0$

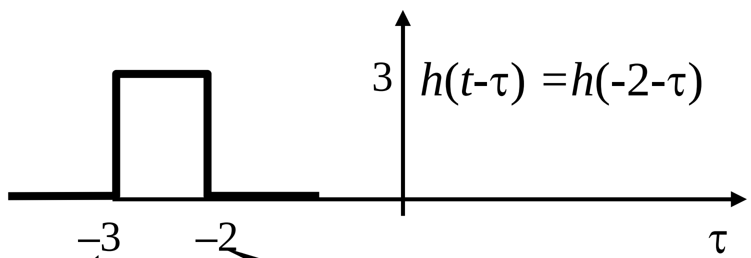
Motivating Step #4: Shift by t to get $h(t-\tau)$ & Its Edges

Just looking at 2 "arbitrary" t values

In Each Case We Get

$$\tau_{L,t} = t + \tau_{L,0}$$
$$\tau_{R,t} = t + \tau_{R,0}$$

For t = -2



$\tau_{L,t} = t + \tau_{L,0}$

$\tau_{L,t} = t - 1$

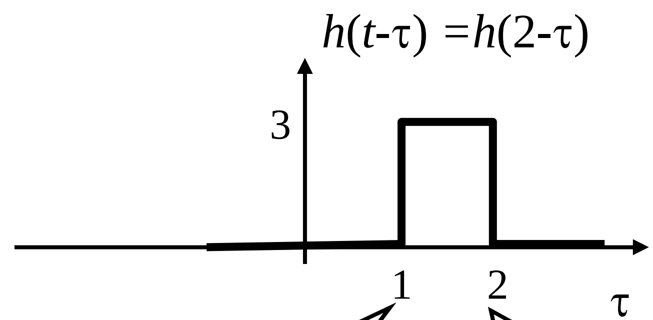
$\tau_{L,-2} = -2 - 1$

$\tau_{R,t} = t + \tau_{R,0}$

$\tau_{R,t} = t + 0$

$\tau_{R,-2} = -2 + 0$

For t = 2



$\tau_{L,t} = t + \tau_{L,0}$

$\tau_{L,t} = t - 1$

$\tau_{L,2} = 2 - 1$

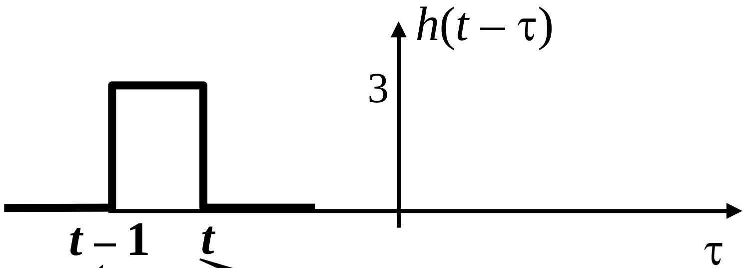
$\tau_{R,t} = t + \tau_{R,0}$

$\tau_{R,t} = t + 0$

$\tau_{R,2} = 2 + 0$

Doing Step #4: Shift by *t* to get *h*(*t*-τ) & Its Edges

For *Arbitrary* Shift by *t*



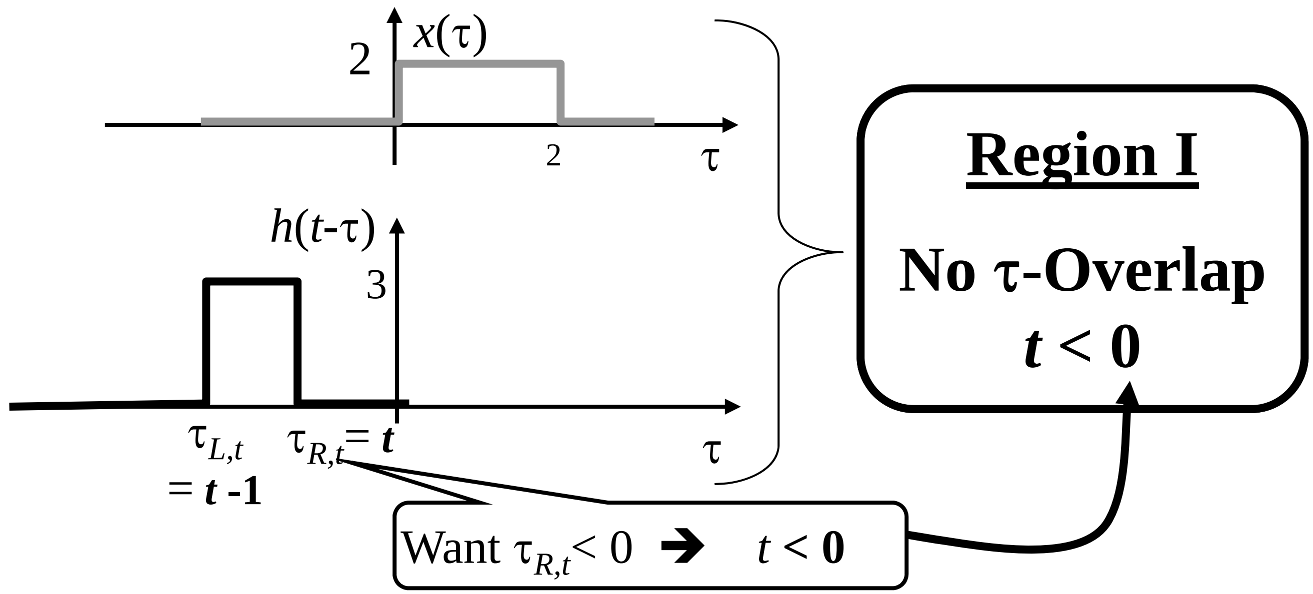
$\tau_{L,t} = \boldsymbol{t} + \tau_{L,0}$

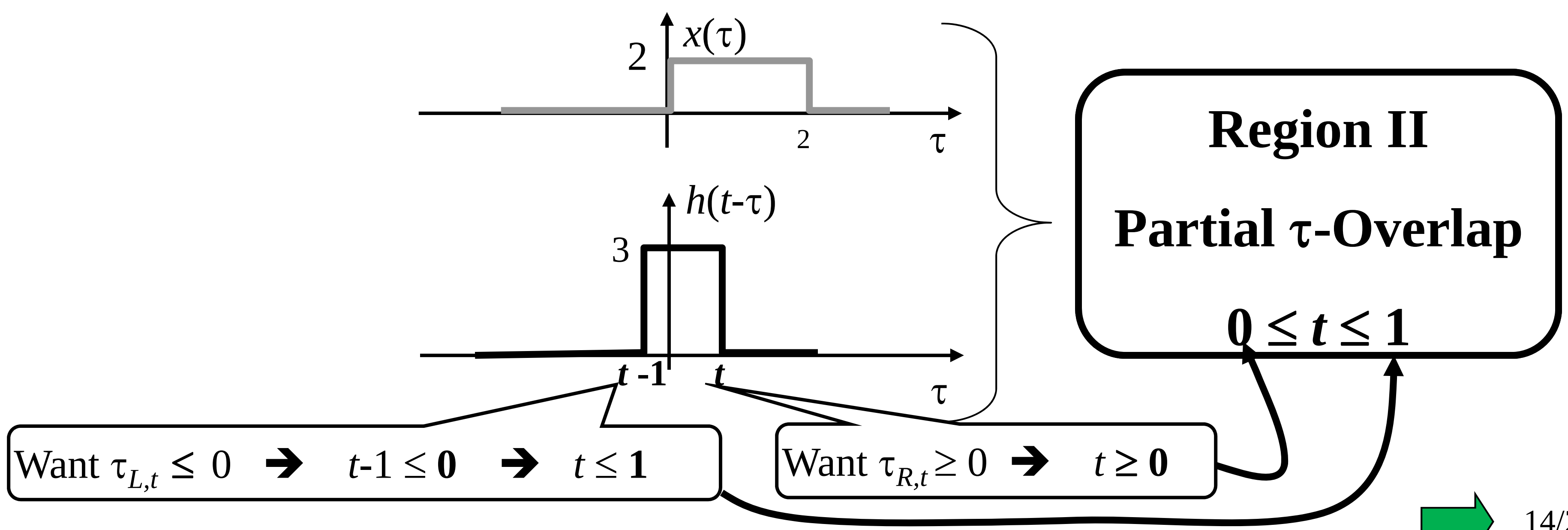
$\tau_{L,t} = \boldsymbol{t} - \mathbf{1}$

$\tau_{R,t} = \boldsymbol{t} + \tau_{R,0}$

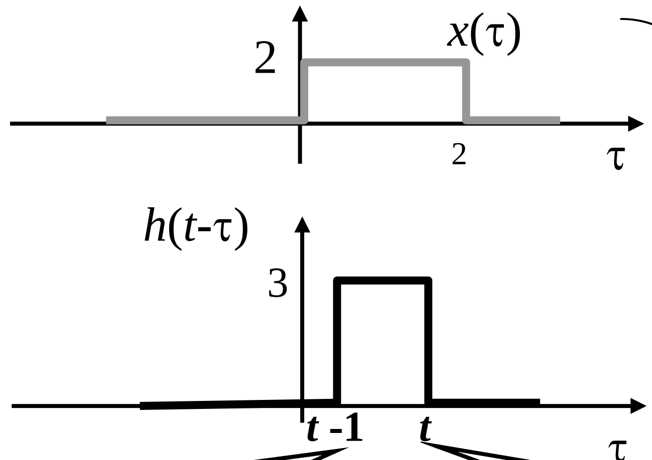
$\tau_{R,t} = \boldsymbol{t} + \mathbf{0}$

Step #5: Find Regions of τ-Overlap





Step #5 (Continued): Find Regions of τ-Overlap



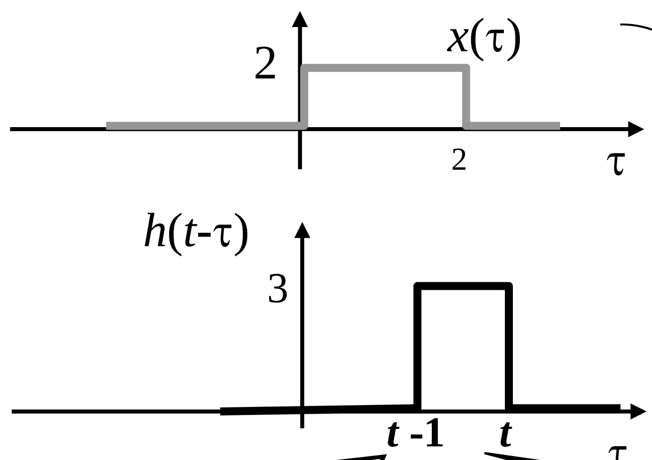
Region III

Total τ-Overlap

$1 < t \leq 2$

Want $\tau_{L,t} > 0$ ➔ t-1 > **0** ➔ $t > 1$

Want $\tau_{R,t} \leq 2$ ➔ $t \leq 2$



Region IV

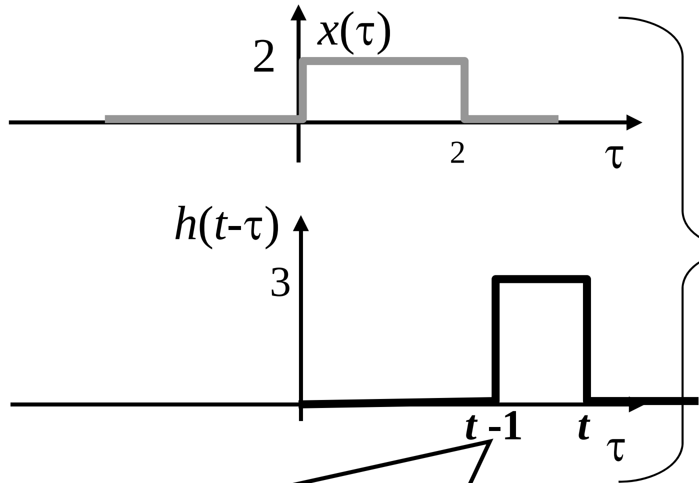
Partial τ-Overlap

$2 < t \leq 3$

Want $\tau_{L,t} \leq 2$ ➔ t-1 ≤ 2 ➔ $t \leq 3$

Want $\tau_{R,t} > 2$ ➔ $t > 2$

Step #5 (Continued): Find Regions of τ-Overlap

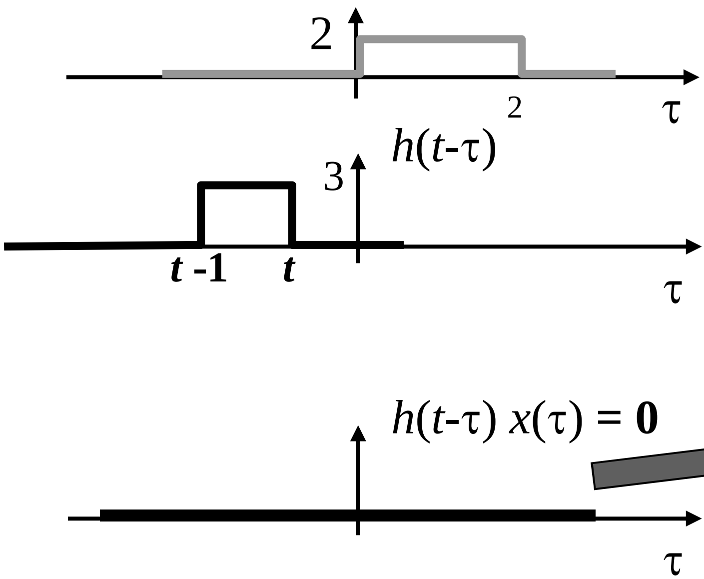


Region V

No τ-Overlap
$t > 3$

Want $\tau_{L,t} > 2$ ➔ $t-1 > 2$ ➔ $t > 3$

Step #6: Form Product & Integrate For Each Region



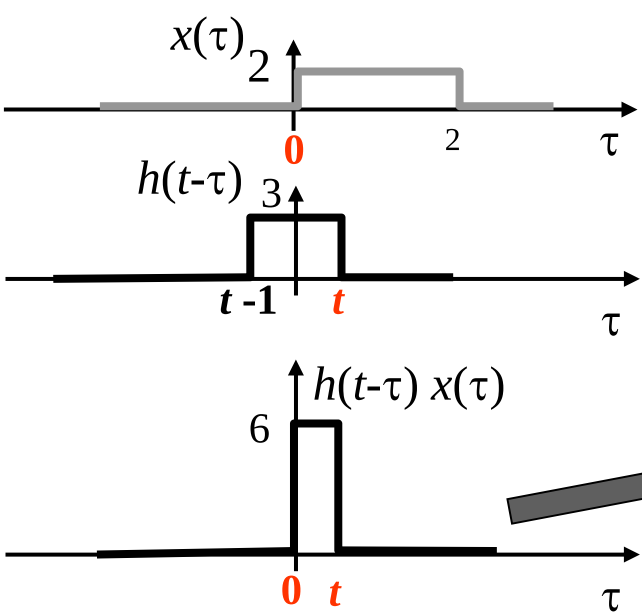
Region I: *t* < 0

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$= \int_{-\infty}^{\infty} 0 d\tau = 0$$

$$y(t) = 0 \quad \text{for all} \quad t < 0$$

With 0 integrand the limits don't matter!!!



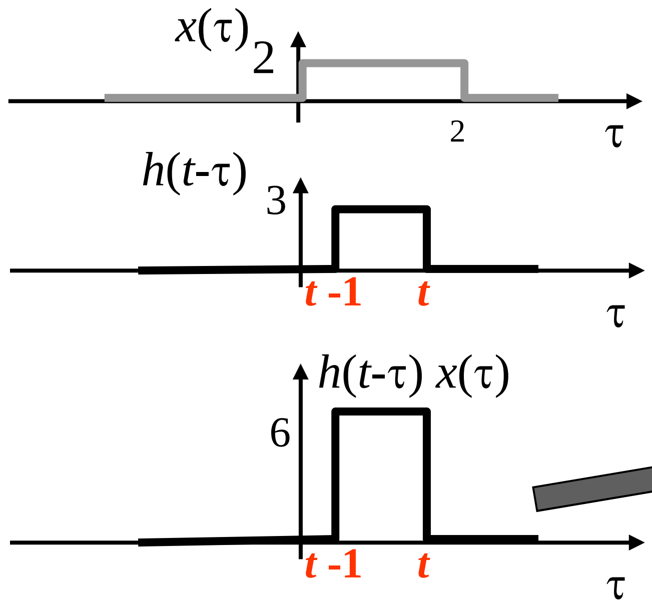
Region II: 0 ≤ *t* ≤ 1

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$= \int_{0}^{t} 6 d\tau = \left[6\tau\right]_0^t = 6t - 6\times 0 = 6t$$

$$y(t) = 6t \quad \text{for} \quad 0 \le t \le 1$$

Step #6 (Continued): Form Product & Integrate For Each Region

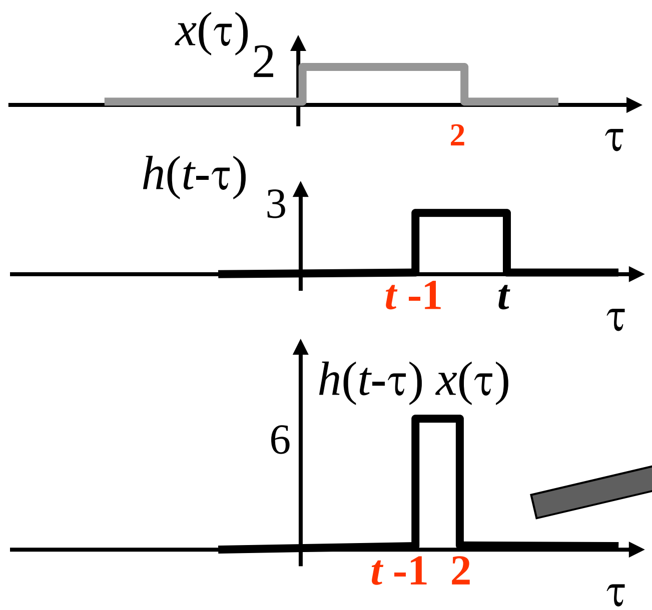


Region III: $1 < t \leq 2$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$= \int_{t-1}^{t} 6d\tau = [6\tau]_{t-1}^{t} = 6t - 6(t-1) = 6$$

$y(t) = 6$ for all t such that: $1 < t \leq 2$



Region IV: $2 < t \leq 3$

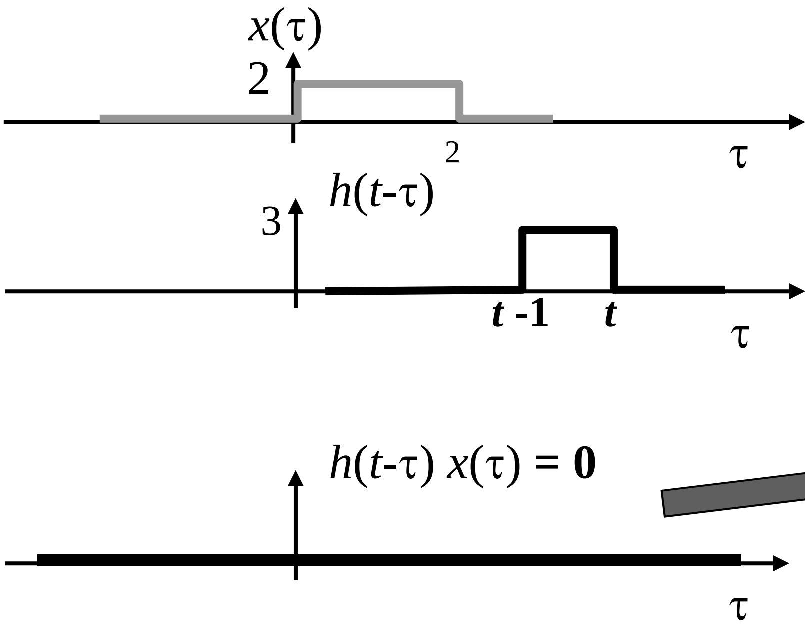
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$= \int_{t-1}^{2} 6d\tau = [6\tau]_{t-1}^{2} = 6 \times 2 - 6(t-1) = -6t + 18$$

$y(t) = -6t + 18$ for $2 < t \leq 3$

Step #6 (Continued): Form Product & Integrate For Each Region

Region V: $t > 3$



$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$= \int_{-\infty}^{\infty} 0 d\tau = 0$$

$$y(t) = 0 \quad \text{for all} \quad t > 3$$

Step #7: "Assemble" Output Signal

Region I	Region II	Region III	Region IV	Region V
$t < 0$	$0 \le t \le 1$	$1 < t \le 2$	$2 < t \le 3$	$t > 3$
$y(t) = 0$	$y(t) = 6t$	$y(t) = 6$	$y(t) = -6t + 18$	$y(t) = 0$