BINGHAMTON UNIVERSITY
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EECE 301
Signals & Systems
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Note Set #32

- D-T Systems: Z-Transform … Solving Difference Eqs. w/ ICs.

Two Different Scenarios for ZT Analysis

We've already used the ZT to analyze a DT system described by a Difference Equation…

However, our focus there was:

- For inputs that *could* exist for all time: $-\infty < n < \infty$
- For systems that did not have Initial Conditions

Can't really think of ICs if the signal never really "starts"…

This is a common view in areas like signal processing and communications…

For that we used the bilateral ZT and found: $y[n] = Z^{-1}\{H(z)X(z)\}$

But in some areas (like control systems) it is more common to consider:

- Inputs that *Start* at time $n = 0$ (input $x[n] = 0$ for $n < 0$)
- Systems w/ Initial Conditions (output $y[n]$ has values for some $n < 0$)

For that scenario it is best to use the unilateral ZT…

One sided Z-transform

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

z is complex-valued

Note that we will apply this to $x[n]$ even though it has non-zero ICs

Properties of Unilateral ZT

Most of the properties are the same as for the bilateral form

But… there are important difference for the unilateral ZT of shifted signals

Unilateral ZT of Right Shift for **Causal Signal**

Let $x[n] = 0, \; n < 0$

We use the symbol for an input here since we now assume our input $x[n]$ to be **causal**.

$$\text{If } x[n] \leftrightarrow X(z), \quad \text{then} \quad x[n-q] \leftrightarrow z^{-q}X(z)$$

$$\text{"Proof"}: \quad X(z) = x[0]z^{0} + x[1]z^{-1} + x[2]z^{-2} + \ldots$$

$$Z\{x[n-q]\} = \underbrace{0z^{0} + 0z^{-1} + \ldots + 0z^{-q+1}}_{=0} + x[0]z^{-q} + x[1]z^{-q-1} + \ldots$$

$$= x[0]z^{0}z^{-q} + x[1]z^{-1}z^{-q} + x[2]z^{-2}z^{-q} + \ldots$$

Pull out the z^{-q}

$$= z^{-q}\underbrace{\left[x[0]z^{0} + x[1]z^{-1} + \ldots\right]}_{= X(z)}$$

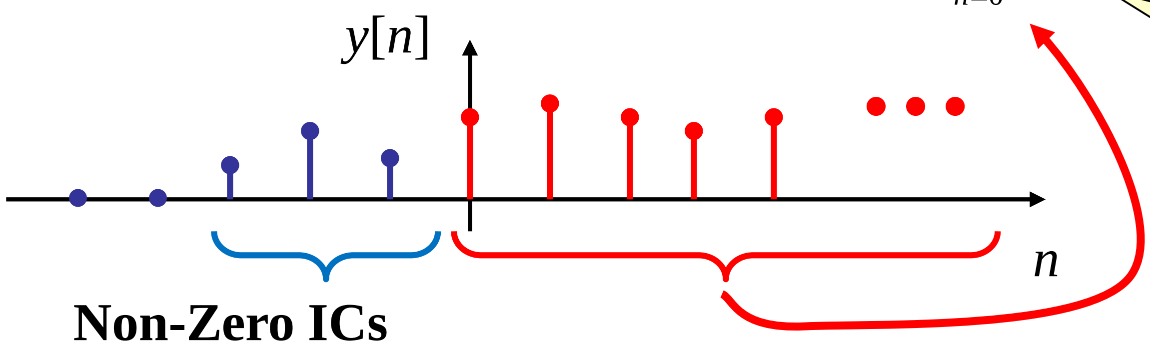
Unilateral ZT of Right Shift for **Non-Causal Signal**

We use the symbol for output here since we now assume our output $y[n]$ to be **non-causal**.

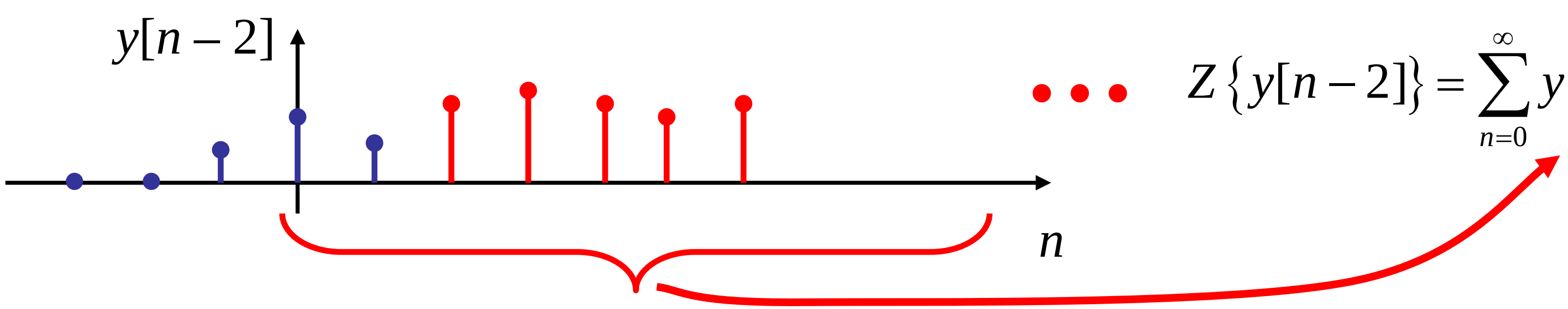
Let $y[n]$ be a non-causal signal… $y[n] \neq 0$ for some $n < 0$

Then the One-Sided ZT is: $y[n] \quad \leftrightarrow \quad Y(z) = \sum_{n=0}^{\infty} y[n] z^{-n}$

Because y[n] is not causal… not all non-zero values of $y[n]$ are used here!!!



Note that right-shifting a non-causal signal brings new values into the one-sided ZT summation!!!



$$Z\{y[n-2]\} = \sum_{n=0}^{\infty} y[n-2] z^{-n}$$

What is $Z\{y[n-q]\}$ in terms of $Y(z)$??

We'll write this property for the first 2 values of q…

$$y[n-1] \quad \leftrightarrow \quad z^{-1}Y(z) + y[-1]$$
$$y[n-2] \quad \leftrightarrow \quad z^{-2}Y(z) + y[-1]z^{-1} + y[-2]$$
$$\vdots \qquad\qquad \vdots$$

… and then write the general result:

$$y[n-q] \quad \leftrightarrow \quad z^{-q}Y(z) + y[-1]z^{-q+1} + y[-2]z^{-q+2} + \ldots + z^{-1}y[-q+1] + y[-q]$$

"Proof" for $q = 2$

$$Z\{y[n-q]\} = y[-2]z^{0} + y[-1]z^{-1} \quad + \quad y[0]z^{-2} + y[1]z^{-3} + \ldots$$

$$= y[-2]z^{0} + y[-1]z^{-1} \quad + \quad z^{-2}\left(y[0]z^{0} + y[1]z^{-1} + \ldots\right)$$

Parts that get "shifted into" the one-sided ZT's "machinery"

$Y(z)$

Now… we've got all the ZT machinery needed to solve a D.E. with ICs!!!

Solving a First-order Difference Equation using the ZT

Given: $y[n]+ay[n-1]=bx[n]$

$\text{IC}=y[-1]$

$x[n]$ for $n=0, 1, 2, \ldots$

Solve for: $y[n]$ for $n = 0, 1, 2, \ldots$

Recalling recursive form:

$$y[n]=ay[n-1]+bx[n]$$

we see why one IC is needed!

Take ZT of difference equation:

$$Z\{y[n]+ay[n-1]\}=Z\{bx[n]\}$$

Use Linearity of ZT

$$Z\{y[n]\}+aZ\{y[n-1]\}=bZ\{x[n]\}$$

$Z\{y[n]\}$: $Y(z)$

$Z\{x[n]\}$: $X(z)$

Need Right-Shift Property… for non-causal signal because $y[n]$ has ICs!!!

$$Z\{y[n-1]\}=z^{-1}Y(z)+y[-1]$$

Using these results gives:

$$Y(z)+a\left[z^{-1}Y(z)+y[-1]\right]=bX(z)$$

…which is an algebraic equation that can be solved for $Y(z)$!

Now…Solving that algebraic equation $Y(z)$ gives:

Not the best form for doing Inverse ZT… we want things in terms of z not z^{-1}

$$Y(z) = \frac{-ay[-1]}{1+az^{-1}} + \frac{b}{1+az^{-1}} X(z)$$

Multiply each term by z/z

$$Y(z) = -ay[-1]\frac{z}{z+a} + \frac{bz}{z+a} X(z)$$

On ZT Table

Part due to input signal modified by **Transfer Function**

$$H(z) = \frac{bz}{z+a}$$

$$y[n] = -ay[-1](-a)^n u[n] + Z^{-1}\{H(z)X(z)\}$$

If $|a| < 1$ this dies out as $n \uparrow$, its an IC-driven transient

If the ICs are zero, this is all we have!!!

Two parts to the solution: one due to ICs and one due to Input!!!

Ex.: Solving a Difference Equation using ZT: 1st-Order System w/ Step Input

For $x[n] = u[n] \quad \leftrightarrow \quad X(z) = \dfrac{z}{z-1}$

Then using our general results we just derived we get:

$$Y(z) = \frac{-ay[-1]z}{z+a} + \left(\frac{bz}{z+a}\right)\left(\frac{z}{z-1}\right)$$

For now assume that $a \neq -1$ so we don't have a repeated root.

Then doing Partial Fraction Expansion we get (and we have to do the PFE by hand because we don't know a… but it is not that hard!!!)

$$Y(z) = \frac{-ay[-1]z}{z+a} + \frac{\left(\frac{ab}{a+1}\right)z}{z+a} + \frac{\left(\frac{b}{a+1}\right)z}{z-1}$$

Now using ZT Table we get:

$$y[n] = -ay[-1](-a)^n + \frac{b}{a+1}\left[a(-a)^n + (1)^n\right] \qquad n = 0, 1, 2, \ldots$$

IC-Driven Transient: decays if system is stable

Input-Driven Output… 2 Terms:

1st term decays (called "Transient")
2nd term persists (called "Steady State")

Solving a Second-order Difference Equation using the ZT

The Given Difference Equation: $y[n]+a_1 y[n-1]+a_2 y[n-2]=b_0 x[n]+b_1 x[n-1]$

Assume that the input is causal

Assume you are given ICs: y[-1] & y[-2]

Find the system response $y[n]$ for n = 0, 1, 2, 3, …

Take the ZT using the non-causal right-shift property:

$$Y(z)+a_1\left(z^{-1}Y(z)+y[-1]\right)+a_2\left(z^{-2}Y(z)+z^{-1}y[-1]+y[-2]\right)$$
$$=b_0 X(z)+b_1 z^{-1}X(z)$$

Errors in Video!

$$Y(z)=\frac{-(a_1 y[-1]+a_2 y[-2])z^{-1}-a_2 y[-1]}{1+a_1 z^{-1}+a_2 z^{-2}}+\underbrace{\frac{b_0+b_1 z^{-1}}{1+a_1 z^{-1}+a_2 z^{-2}}}_{H(z)}X(z)$$

Due to IC's… decays if system is stable

Due to input – will have transient part and steady-state part

Let's take a look at the IC-Driven transient part:

Errors in Video!

$$Y_{zi}(z) = \frac{-(a_1 y[-1] + a_2 y[-2])z^{-1} - a_2 y[-1]}{1 + a_1 z^{-1} + a_2 z^{-2}} = \frac{A - Bz^{-1}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

Multiply top and bottom by z^2:

$$Y_{zi}(z) = \frac{Az^2 + Bz}{z^2 + a_1 z + a_2}$$

Now to do an inverse ZT on this requires a bit of trickery…
Take the bottom two entries on the ZT table and form a linear combination:

$$C_1 a^n \cos(\Omega_o n)u[n] + C_2 a^n \sin(\Omega_o n)u[n] \leftrightarrow \frac{C_1 z^2 + a\left(C_2 \sin(\Omega_o) - C_1 \cos(\Omega_o)\right)z}{z^2 - 2a\cos(\Omega_o)z + a^2}$$

$$a = \sqrt{a_2} \qquad \Omega_0 = \cos^{-1}\left[\frac{-a_1}{2\sqrt{a_2}}\right]$$

$$C_1 = A \qquad C_2 = \frac{B}{a\sin(\Omega_0)} - C_1 \frac{\cos(\Omega_0)}{\sin(\Omega_0)}$$

Compare & Identify

Finally, by a trig ID we know that

$$C_1 a^n \cos(\Omega_o n)u[n] + C_2 a^n \sin(\Omega_o n)u[n] = Ca^n \cos(\Omega_o n + \theta)u[n]$$

So… all of this machinery leads to the insight that the IC-Driven transient of a second-order system will look like this:

$$y_{zi}[n] = Ca^n \cos(\Omega_o n + \theta)u[n]$$

…where:

1. The <u>frequency Ω_0</u> and <u>exponential a</u> are set by the Characteristic Eq.
2. The <u>amplitude C</u> and the <u>phase θ</u> are set by the ICs

$$a = \sqrt{a_2} \qquad \Omega_0 = \cos^{-1}\left[\frac{-a_1}{2\sqrt{a_2}}\right]$$

Note: If $|a_2| < 1$ then we get a <u>decaying</u> response!!

Solving a N^th-order Difference Equation using the ZT

$$y[n]+\sum_{i=1}^{N} a_i y[n-i] = \sum_{i=0}^{M} b_i x[n-1]$$

Contains $x[n]$, $x[n\text{-}1]$,…

If this system is causal, we won't have $x[n+1]$, $x[n+2]$, etc. here

$$A(z) = z^N + a_1 z^{N-1} + \ldots + a_{N-1} z + a_N$$

$$B(z) = b_0 z^N + b_1 z^{N-1} + \ldots + b_M z^{N-M}$$

$$C(z) = \text{depends on the IC's}$$

Transforming gives:

$$Y(z) = \frac{C(z)}{A(z)} + \frac{B(z)}{A(z)} X(z)$$

$\frac{B(z)}{A(z)}$: $H(z)$ – transfer function

$\frac{B(z)}{A(z)}X(z)$: **$Y_{zs}(z)$ – "Zero State Part"**

$\frac{C(z)}{A(z)}$: **$Y_{zi}(z)$ – "Zero Input Part"**

$A(z)$ is denominator of Transfer Function… "Characteristic Poly."

Interpreting the General Output Result $Y(z) = Y_{zi}(z) + Y_{zs}(z)$

$Y_{zi}(z)$ Zero-Input Response: Is due to ICs… and its nature is defined by $A(z)$!

$$Y_{zi}(z) = \frac{C(z)}{A(z)} = \frac{k_1 z}{z - p_1} + \frac{k_2 z}{z - p_2} + \cdots + \frac{k_N z}{z - p_N}$$

For simplicity… assumed distinct poles

$$y_{zi}[n] = k_1 p_1^n u[n] + k_2 p_2^n u[n] + \cdots + k_N p_N^n u[n]$$

Decays if $|p_i| < 1$

System Poles …roots of $A(z)$… play big role here!!!

$Y_{zs}(z)$ Zero-State Response: Is due to input & its nature is defined by $A(z)$ *and* $X(z)$

For simplicity assume $X(z) = E(z)/F(z)$

$$Y_{zs}(z) = \frac{B(z)}{A(z)} \frac{E(z)}{F(z)} = \frac{c_1 z}{z - p_1} + \frac{c_2 z}{z - p_2} + \cdots + \frac{c_N z}{z - p_N} + \frac{D(z)}{F(z)}$$

Some Polynomial that "falls out of" PFE

$$y_{zs}[n] = c_1 p_1^n u[n] + c_2 p_2^n u[n] + \cdots + c_N p_N^n u[n] + y_{ss}[n]$$

ZS Transient Response…Decays if $|p_i| < 1$

ZS Steady State Response

So… the output of a stable, causal Difference Equation with ICs and a causal input is….

$$y[n] = y_{zi}[n] + \left[y_{zs,tr}[n] + y_{zs,ss}[n] \right]$$

System Poles …roots of *A*(z)… play big role here!!!

Might decay but might not… depends on interaction of system and input

Both decay if system is stable!