BINGHAMTON UNIVERSITY
State University of New York

EECE 301
Signals & Systems
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Note Set #26

- D-T Systems: Transfer Function and Frequency Response

Finding the Transfer Function from Difference Eq.

Recall: we found a DT system's freq. resp. $H(\Omega)$ by analyzing for input $e^{j\Omega n}$ or by taking DTFT of the Diff Eq. Here we can either analyze the system for input z^n or take the ZT of the Diff Eq… here we do the later:

$$y[n]+a_1 y[n-1]+\ldots+a_N y[n-N]=b_0 x[n]+b_1 x[n-1]+\ldots+b_M x[n-M]$$

ZT →

$$ZT\left\{y[n]+a_1 y[n-1]+\ldots+a_N y[n-N]\right\} = ZT\left\{b_0 x[n]+b_1 x[n-1]+\ldots+b_M x[n-M]\right\}$$

Delay Prop. & Algebra →

$$Y(z)\left[1+a_1 z^{-1}+\ldots+a_N z^{-N}\right] = X(z)\left[b_0+b_1 z^{-1}+\ldots+b_M z^{-M}\right]$$

Algebra →

$$Y(z)=\underbrace{\left[\frac{b_0+b_1 z^{-1}+\ldots+b_M z^{-M}}{1+a_1 z^{-1}+\ldots+a_N z^{-N}}\right]}_{H(z)} X(z)$$

So… can just write H(z) by inspection of D.E. coefficients!

From this $H(z)$ we know how to compute the output: y = filter(b,a,x);

Poles and Zeros of Transfer Function

$$H(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{1 + a_1 z^{-1} + \ldots + a_N z^{-N}} \Rightarrow H(z) = z^{(N-M)} \frac{b_0 z^M + b_1 z^{M-1} + \ldots + b_M}{z^N + a_1 z^{N-1} + \ldots + a_N}$$

Define the ***polynomials*** $A(z)$ and $B(z)$ so that: $$H(z) = z^{(N-M)} \frac{B(z)}{A(z)}$$

Assume there are no common roots in the numerator $B(z)$ and denominator $A(z)$.
(If not, assume they've been cancelled and redefine $B(z)$ and $A(z)$ accordingly)

Poles of *H(z)*: The values on the complex z-plane where $|H(z)| \to \infty$

Zeros of *H(z)*: The values on the complex z-plane where $|H(z)| = 0$

The roots of the **denominator** polynomial $A(z)$ determine N poles.

The roots of the **Numerator** polynomial $B(z)$ determine M zeros.

The term $z^{(N-M)}$ gives poles/zeros at the origin according to:

- If $N > M$: $N - M$ **zeros @ Origin**
- If $N < M$: $M - N$ **poles @ Origin**

Example: Finding Poles and Zeros

$$y[n]-\tfrac{1}{\sqrt{2}}y[n-1]+\tfrac{1}{4}y[n-2]=2x[n]+x[n-1]$$

$$H(z)=\frac{2+z^{-1}}{1-\tfrac{1}{\sqrt{2}}z^{-1}+\tfrac{1}{4}z^{-2}}$$

$$H(z)=\frac{z(2z+1)}{z^{2}-\tfrac{1}{\sqrt{2}}z+\tfrac{1}{4}}$$

$$\text{zeros: } z=0,\ z=-1/2$$

$$\text{poles: } z=\frac{1}{2\sqrt{2}}(1\pm j)$$

p =1 zero at origin

Conjugate Pair

Using MATLAB:

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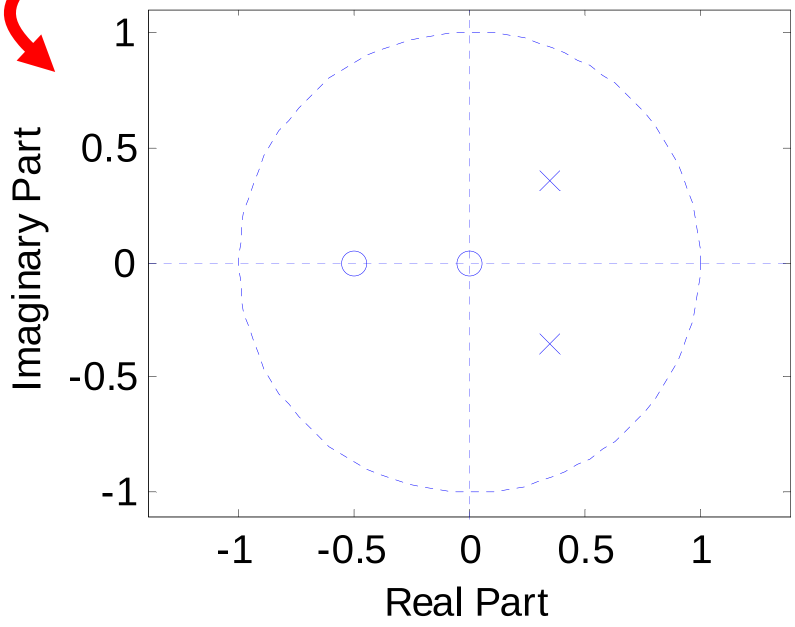
>> zplane([2 1],[1 -(1/sqrt(2)) 1/4])
```

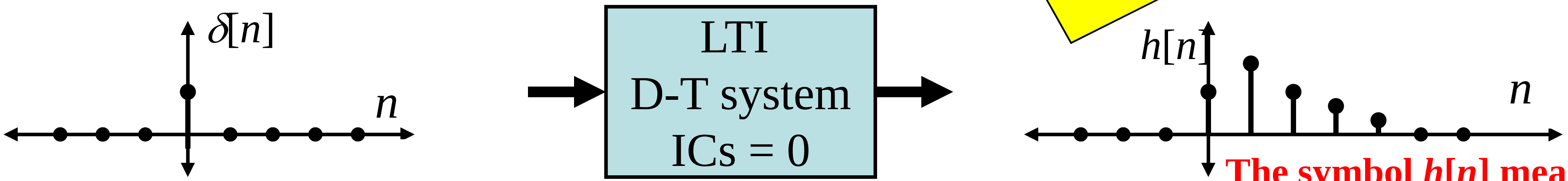
Coeff. Vectors ***MUST*** be **rows**



Impulse Response of System

Sometimes looking at how a system responds to the impulse function (i.e., delta sequence) δ[n] can tell much about a system. Hitting a system with δ[n] is lot like ringing a bell to hear how it sounds…

Note: If system is causal, then $h[n] = 0$ for $n < 0$



The symbol $h[n]$ means "the impulse response".

Noting that the ZT of δ[n] = 1 and using the properties of the transfer function and the Z transform:

$$h[n] = Z^{-1}\{H(z)Z\{\delta[n]\}\}$$

$$h[n] = Z^{-1}\{H(z)\}$$

$$h[n] = IDTFT\{H(\Omega)\}$$

From PFE and Poles/Zeros we see that a TF like this: $H(z) = z^{(N-M)}\dfrac{B(z)}{A(z)}$

…will have an impulse response with terms like this:

$$h[n] = k_1 p_1^n u[n] + k_2 p_2^n u[n] + \cdots + k_N p_N^n u[n]$$

Some simplifying assumptions made here!

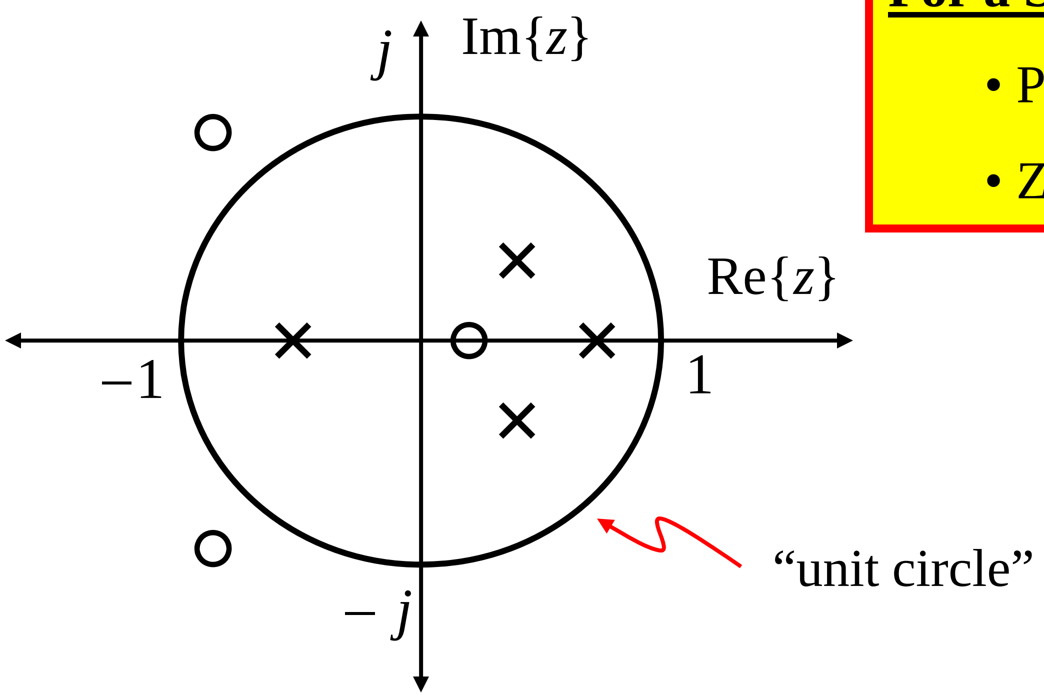
Now… we almost always want this to decay (like a bell!): all poles $|p_i| < 1$

Stability of System

Definition: A system is said to be **stable** if its output will never grow without bound when any bounded input signal is applied… and that seems like a good thing!!!

Without going into all the details… a system with an impulse response that decays "fast enough" is said to be stable.

From our exploration of the effect of poles on the impulse response we say that:



For a Stable System

- Poles must be "inside unit circle"
- Zeros can be anywhere

Relationship: Transfer Function and Freq. Resp.

Recall: DTFT = ZT evaluated on Unit Circle… if UC is inside ROC

Fact: For causal systems UC is inside ROC if all poles are inside UC

$$H(\Omega) = H(z)\big|_{z=e^{j\Omega}}$$

If all poles are inside the UC

We saw how to use freqz before to plot the Frequency Response… this just shows how to plot the Frequency Response from the Transfer Function coefficients:

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-N}}$$

>> num = $[b_0\, b_1 \ldots b_M]$

>> den = $[a_0\, a_1 \ldots a_N]$

must put any zero b_i into the vector

must put any zero a_i into the vector

>> omega = -pi: ?:pi

Pick appropriate spacing

>> H = freqz(num, denom, omega)

>> plot(omega/pi, abs(H))

>> plot(omega/pi, angle(H))

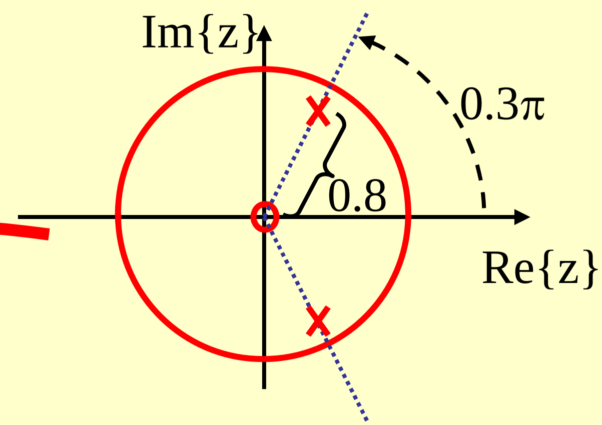
Visualizing Relationship Between TF & FR

Zero @ z = 0

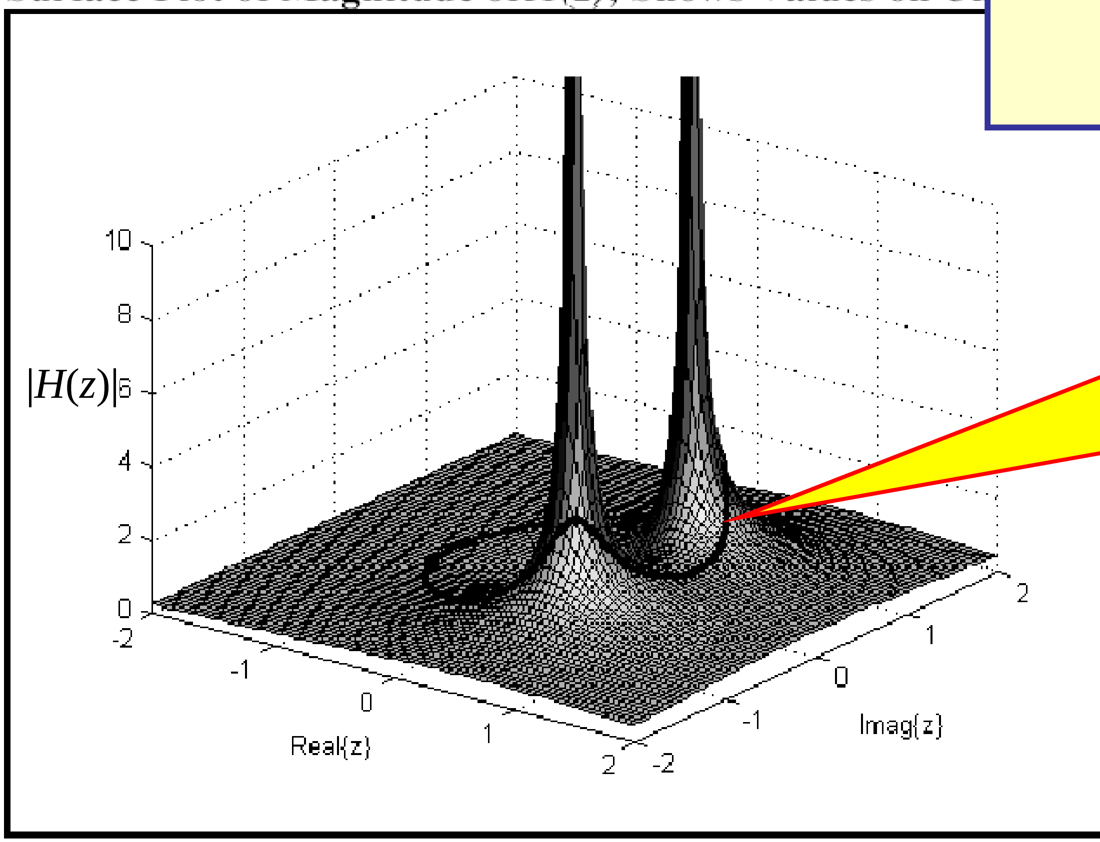
$$H(z) = \frac{z}{\left(z - 0.8e^{j0.3\pi}\right)\left(z - 0.8e^{-j0.3\pi}\right)}$$

Poles @ $z = 0.8e^{\pm j0.3\pi}$

Pole-Zero Plot For This *H*(*z*)



Surface Plot of Magnitude of *H*(*z*); Shows Values on Ur

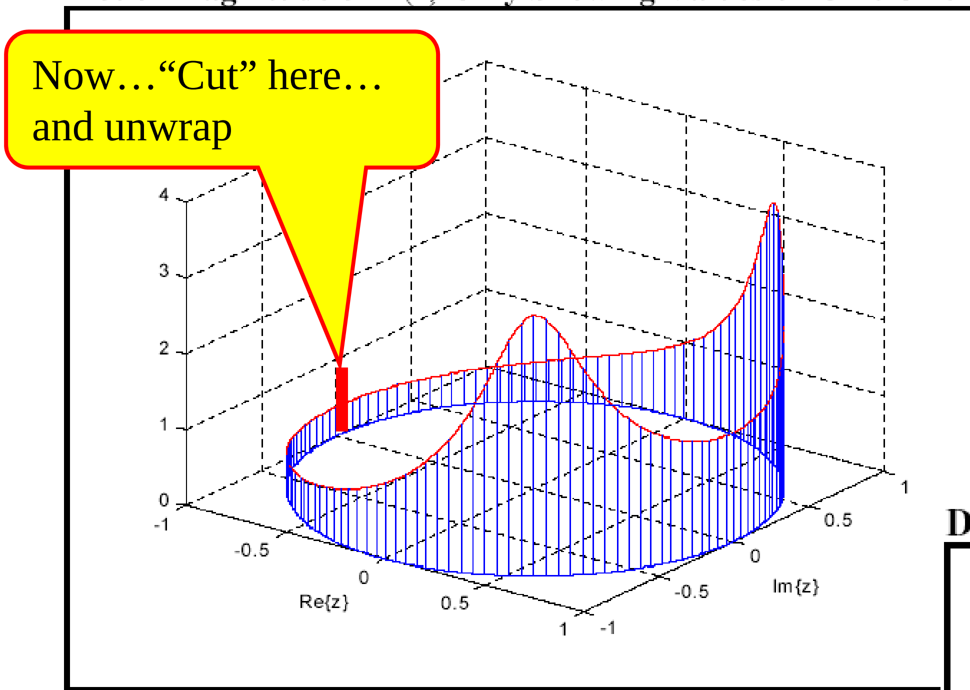


And we know that the Frequency Response is just the Transfer Function evaluated on the Unit Circle.

$$H(\Omega) = H(z)\big|_{z=e^{j\Omega}}$$

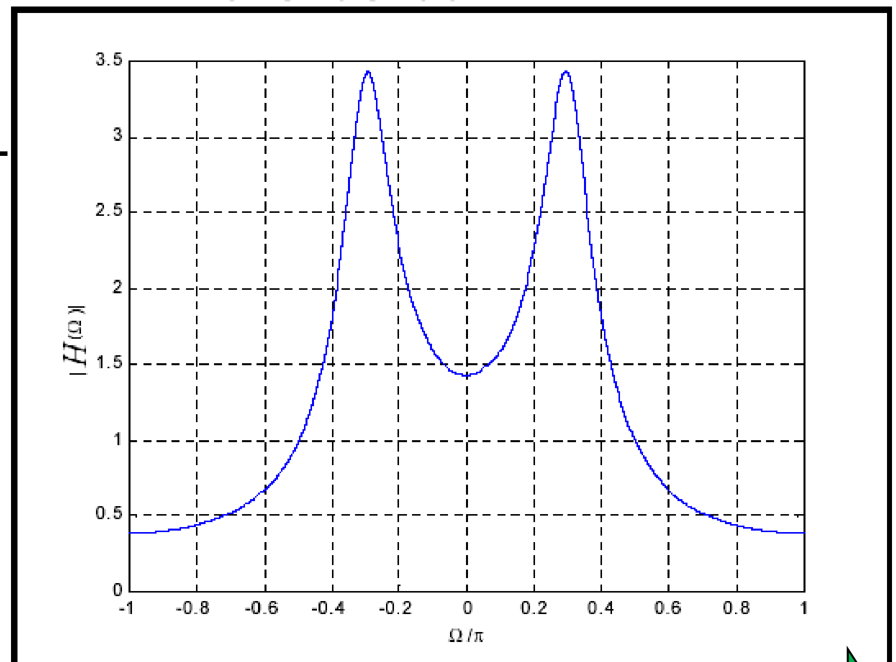
Now… plot just those values on the unit circle:

Plot of Magnitude of *H(z)* Only Showing Values on Unit Circle



This shows after it has been “cut and unwrapped”… and plotted on the Ω axis:

DTFT = ZT on Unit Circle

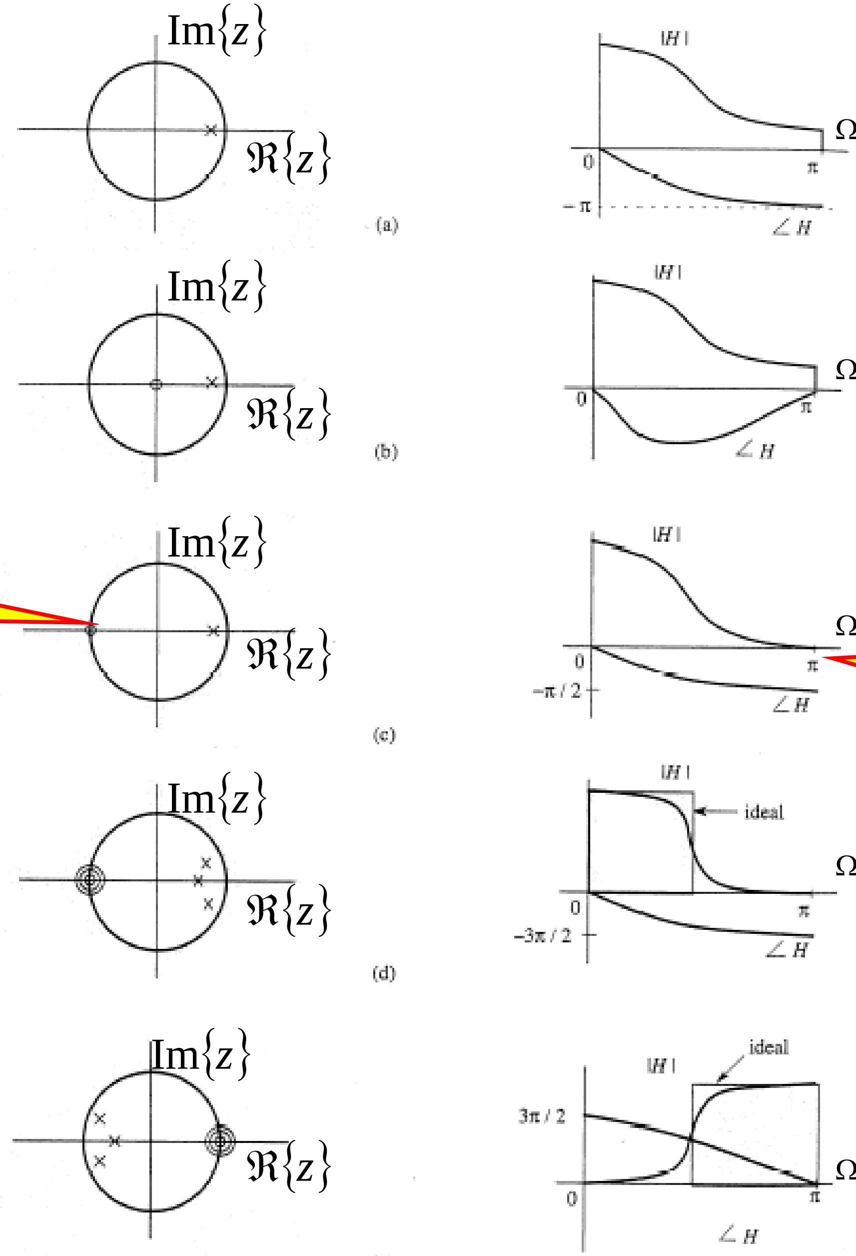


This shows the Frequency Response *H*(Ω) where Ω is the angle around the unit circle… this explains why *H*(Ω) is a periodic function of Ω

Effect of Poles & Zeros on Frequency Response of DT filters

Note: Including a pole or zero at the origin …

…doesn't change the magnitude but does change the phase



Placing a zero at ±π…

…makes |H(π)| = 0

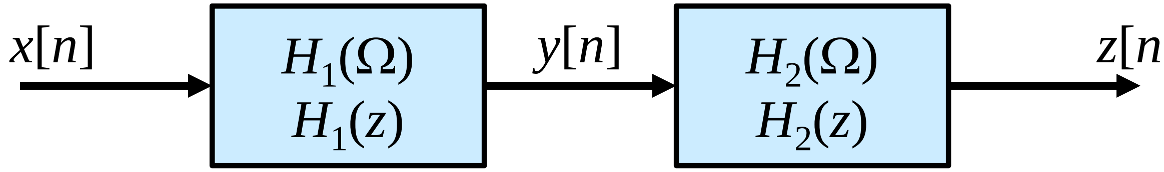
Placing more zeros/poles…

… gives sharper transitions.

Figure from B.P. Lathi, Signal Processing and Linear Systems

Cascade of Systems

Suppose you have a "cascade" of two systems like this:



$$Y(\Omega) = H_1(\Omega)X(\Omega) \qquad Z(\Omega) = H_2(\Omega)Y(\Omega)$$

$$Y(z) = H_1(z)X(z) \qquad Z(z) = H_2(z)Y(z)$$

$$Z(\Omega) = H_2(\Omega)H_1(\Omega)X(\Omega)$$

$$Z(z) = H_2(z)H_1(z)X(z)$$

Thus, the ***overall*** frequency response/transfer function is the product of those of each stage:

$$H_{total}(\Omega) = H_1(\Omega)H_2(\Omega)$$

$$H_{total}(z) = H_1(z)H_2(z)$$

Obviously, this generalizes to a cascade of N systems:

$$H_{total}(\Omega) = H_1(\Omega)H_2(\Omega)\cdots H_N(\Omega)$$

$$H_{total}(z) = H_1(z)H_2(z)\cdots H_N(z)$$