Binghamton University
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EECE 301
Signals & Systems
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Note Set #23

- D-T Systems: DTFT Analysis of DT Systems

Finding the Frequency Response from Difference Eq.

Recall: we found a circuit's freq. resp. $H(\omega)$ by analyzing for input $e^{j\omega t}$

As for a circuit, hypothesize this: $x[n] = e^{j\Omega n} \quad \rightarrow \quad y[n] = H(\Omega)e^{j\Omega n}$

Now sub into this Diff Eq the hypothesized input and output:

$$\boxed{y[n] + a_1 y[n-1] + \ldots + a_N y[n-N] = b_0 x[n] + b_1 x[n-1] + \ldots + b_M x[n-M]}$$

Sub In

$$\begin{aligned} &H(\Omega)e^{j\Omega n} + a_1 H(\Omega)e^{j\Omega(n-1)} + \ldots + a_N H(\Omega)e^{j\Omega(n-N)} \\ &\qquad = b_0 e^{j\Omega n} + b_1 e^{j\Omega(n-1)} + \ldots + b_M e^{j\Omega(n-M)} \end{aligned}$$

Algebra

$$\begin{aligned} &H(\Omega)\cancel{e^{j\Omega n}}\left[1 + a_1 e^{j\Omega(-1)} + \ldots + a_N e^{j\Omega(-N)}\right] \\ &\qquad = \cancel{e^{j\Omega n}}\left[b_0 + b_1 e^{j\Omega(-1)} + \ldots + b_M e^{j\Omega(-M)}\right] \end{aligned}$$

Algebra

$$\boxed{H(\Omega) = \frac{b_0 + b_1 e^{-j\Omega} + \ldots + b_M e^{-j\Omega M}}{1 + a_1 e^{-j\Omega} + \ldots + a_N e^{-j\Omega N}}}$$

So… can just write $H(\Omega)$ by <u>inspection</u> of D.E. coefficients!

DT LTI System Response to a Sinusoid

We've just shown that $x[n] = e^{j\Omega n} \quad \rightarrow \quad y[n] = H(\Omega)e^{j\Omega n}$

By using Euler's formula and linearity we can extend this to:

$$\boxed{x[n] = A\cos(\Omega_o n + \theta) \quad \rightarrow \quad y[n] = |H(\Omega_o)| A\cos(\Omega_o n + \theta + \angle H(\Omega_o))}$$

This tells us that an DT LTI system does two things to a sinusoidal input:

1. It changes its amplitude ***multiplicatively*** with factor $|H(\Omega_o)|$
2. It changes its phase ***additively*** with factor $\angle H(\Omega_o)$

Alternate way to find Frequency Response: Take the DTFT of the Difference Equation and use the Delay Property:

$$\boxed{y[n]+a_1y[n-1]+\ldots+a_Ny[n-N]=b_0x[n]+b_1x[n-1]+\ldots+b_Mx[n-M]}$$

DTFT

$$DTFT\left\{y[n]+a_1y[n-1]+\ldots+a_Ny[n-N]\right\}$$
$$=DTFT\left\{b_0x[n]+b_1x[n-1]+\ldots+b_Mx[n-M]\right\}$$

Delay Prop

$$Y(\Omega)+a_1Y(\Omega)e^{-j\Omega}+\ldots+a_NY(\Omega)e^{-j\Omega N}$$
$$=b_0X(\Omega)+b_1X(\Omega)e^{-j\Omega}+\ldots+b_MX(\Omega)e^{-j\Omega M}$$

Algebra

$$Y(\Omega)\left[1+a_1e^{-j\Omega}+\ldots+a_Ne^{-j\Omega N}\right]$$
$$=X(\Omega)\left[b_0+b_1e^{-j\Omega}+\ldots+b_Me^{-j\Omega M}\right]$$

Algebra

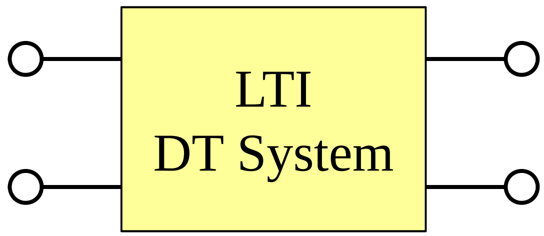
$$Y(\Omega)=\underbrace{\left[\frac{b_0+b_1e^{-j\Omega}+\ldots+b_Me^{-j\Omega M}}{1+a_1e^{-j\Omega}+\ldots+a_Ne^{-j\Omega N}}\right]}_{H(\Omega)}X(\Omega)$$

Same result as on previous page

System analysis via DTFT

Recall the definition of the frequency response:

Input $e^{j\Omega n}$



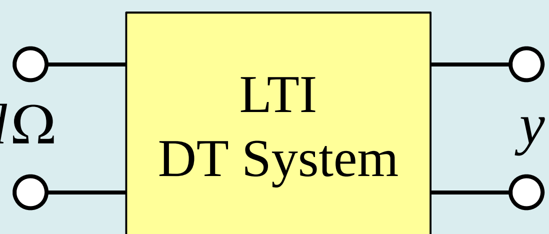
Output $H(\Omega)e^{j\Omega n}$

Input $x[n]$ is a linear combo of sinusoids… the output is a linear combo:

Input

$$x[n] = \int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n}d\Omega$$

Input's DTFT



Output

$$y[n] = \int_{-\pi}^{\pi} H(\Omega)X(\Omega)e^{j\Omega n}d\Omega$$

Output's DTFT

$$\boxed{Y(\Omega) = H(\Omega)X(\Omega)}$$

So we have:

$$x[n] \longrightarrow \boxed{H(\Omega)} \longrightarrow y[n] = \mathscr{F}^{-1}\{Y(\Omega)\}$$

$$X(\Omega) \qquad\qquad \boxed{Y(\Omega) = X(\Omega)H(\Omega)}$$

$$|Y(\Omega)| = |X(\Omega)||H(\Omega)|$$

$$\angle Y(\Omega) = \angle X(\Omega) + \angle H(\Omega)$$

It uses |H(Ω)| to <u>multiplicatively</u> change the amplitude of each input frequency component

It uses ∠H(Ω) to <u>additively</u> change the phase of each input frequency component

So…in general we see that the system frequency response re-shapes the input DTFT's magnitude and phase.

$\Rightarrow$ <u>System can</u>:

-emphasize some frequencies

-de-emphasize other frequencies

Perfectly parallel to the same ideas for CT systems!!!

The above shows how to use DTFT to do general DT system analyses … and it is virtually same as for the CT case!

Two Main Ways to Use Frequency Response for DT LTI Systems

Input: Sinusoid

$$x[n] = A\cos(\Omega_o n + \theta)$$

or

$$x[n] = Ae^{j(\Omega_o n + \theta)}$$

$$y[n] = A\left|H(\Omega_o)\right|\cos(\Omega_o n + \theta + \angle H(\Omega_o))$$

or

$$y[n] = AH(\Omega_o)e^{j(\Omega_o n + \theta)}$$

Error in Video!

Can be applied to each term if input is sum of sinusoids

Error in Video!

Input: Arbitrary

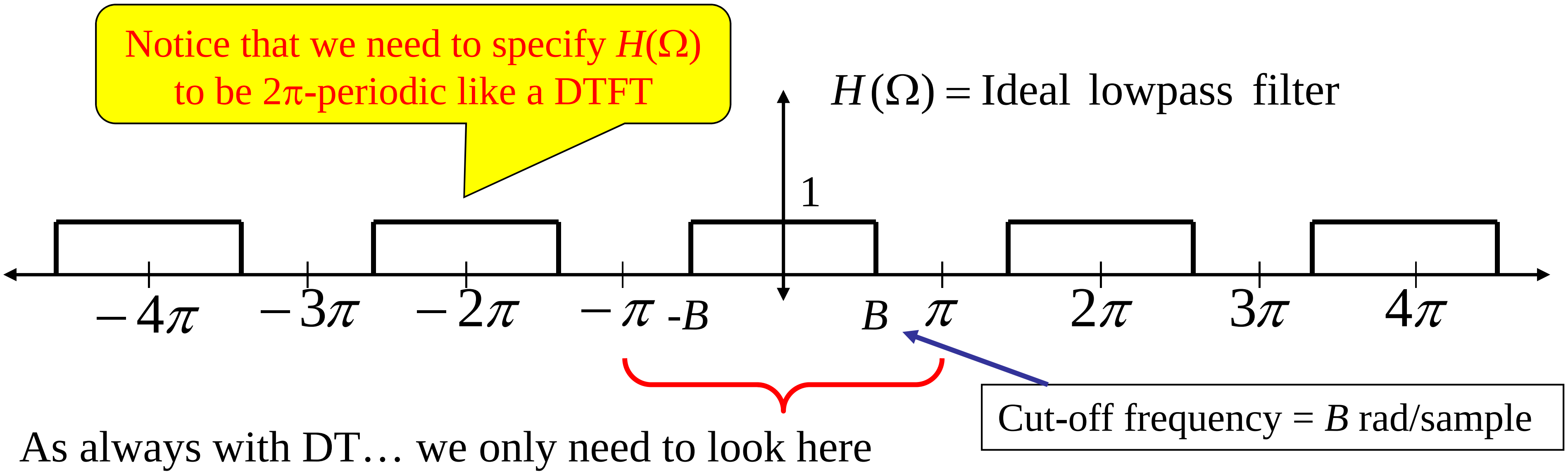
$$x[n] \leftrightarrow X(\Omega)$$

$$y[n] = DTFT^{-1}\left\{H(\Omega)X(\Omega)\right\}$$

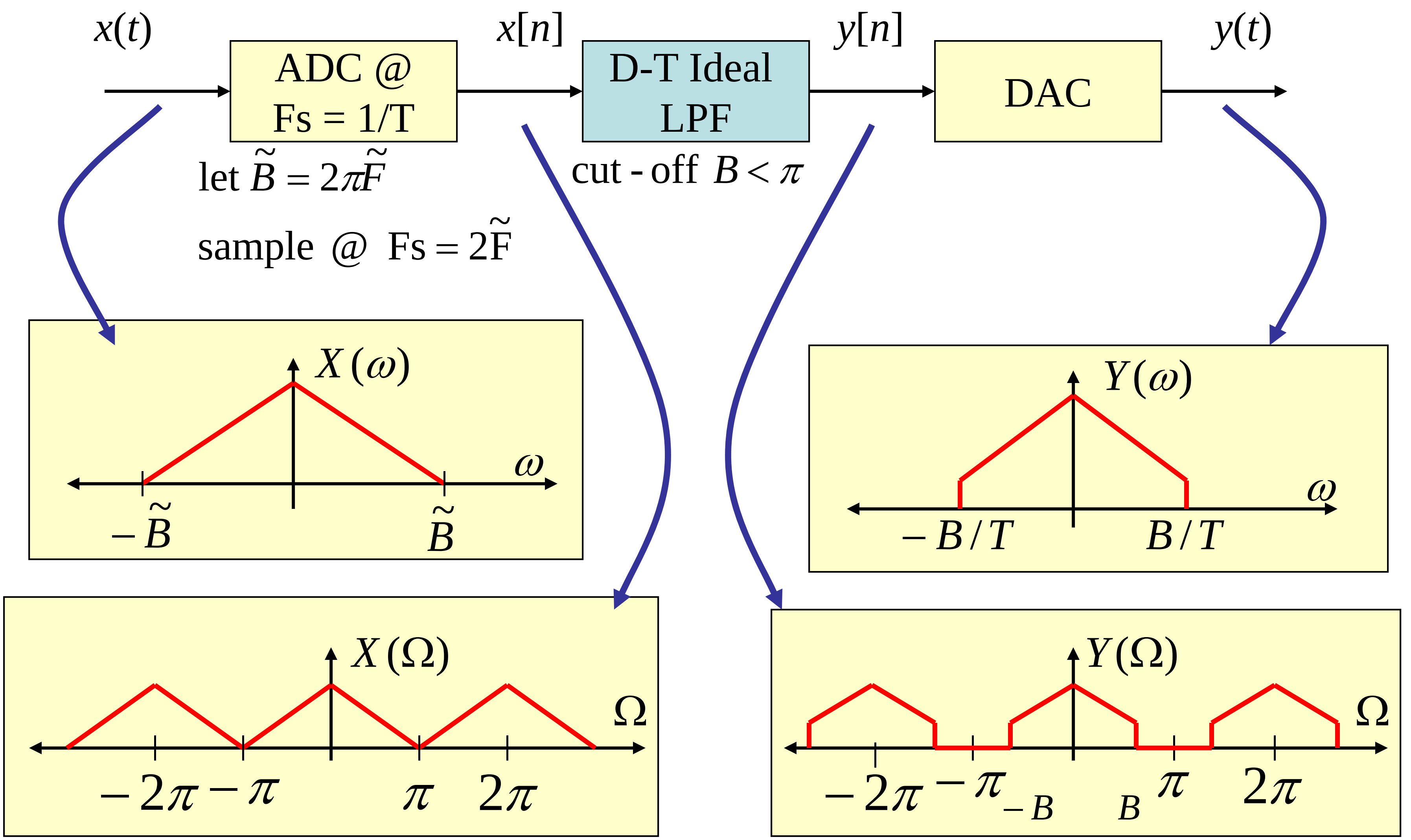
Although we can sometimes actually DO this… we most often use THINK this concept when looking at plot of |H(Ω)| and ∠H(Ω)

Example: “Ideal” D-T lowpass Filter (LPF)

We will see later that we can’t really build such an “ideal” filter but we can strive to get very close…



This slide shows how a DT filter might be employed… but ideal filters can't be built in practice. We'll see later a few practical DT filters.

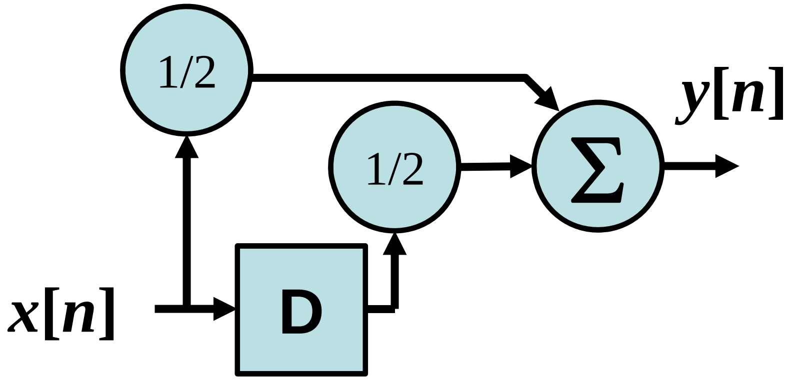


Whole System (ADC – DT filter – DAC) acts like an equiv. C-T system

Example: Simple "Non-Recursive" Filter

Here is a very simple, low quality LPF. Its difference equation and block diagram are:

$$y[n] = \frac{1}{2}x[n] + \frac{1}{2}x[n-1]$$



The general results for Diff Eq & Freq Response are:

$$\boxed{y[n] + a_1 y[n-1] + \ldots + a_N y[n-N] = b_0 x[n] + b_1 x[n-1] + \ldots + b_M x[n-M]}$$

$$\boxed{H(\Omega) = \frac{b_0 + b_1 e^{-j\Omega} + \ldots + b_M e^{-j\Omega M}}{1 + a_1 e^{-j\Omega} + \ldots + a_N e^{-j\Omega N}}}$$

Note that the given filter has none of the so-called feedback terms… such a filter is called a non-recursive filter.

Using the general result for this filter gives:

$$\boxed{H(\Omega) = \frac{1}{2}\left[1 + e^{-j\Omega}\right]}$$

Now, to see what this looks like we find its magnitude….

$$H(\Omega) = \frac{1}{2}\left[1 + \boxed{e^{-j\Omega}}\right]$$

Euler!

$$= \frac{1}{2}\left[(1 + \boxed{\cos(\Omega)) - j\sin(\Omega)}\right]$$

It is now in rect. form…

$$|H(\Omega)| = \sqrt{\left[\tfrac{1}{2}(1+\cos(\Omega))\right]^2 + \left(-\tfrac{1}{2}\sin(\Omega)\right)^2}$$

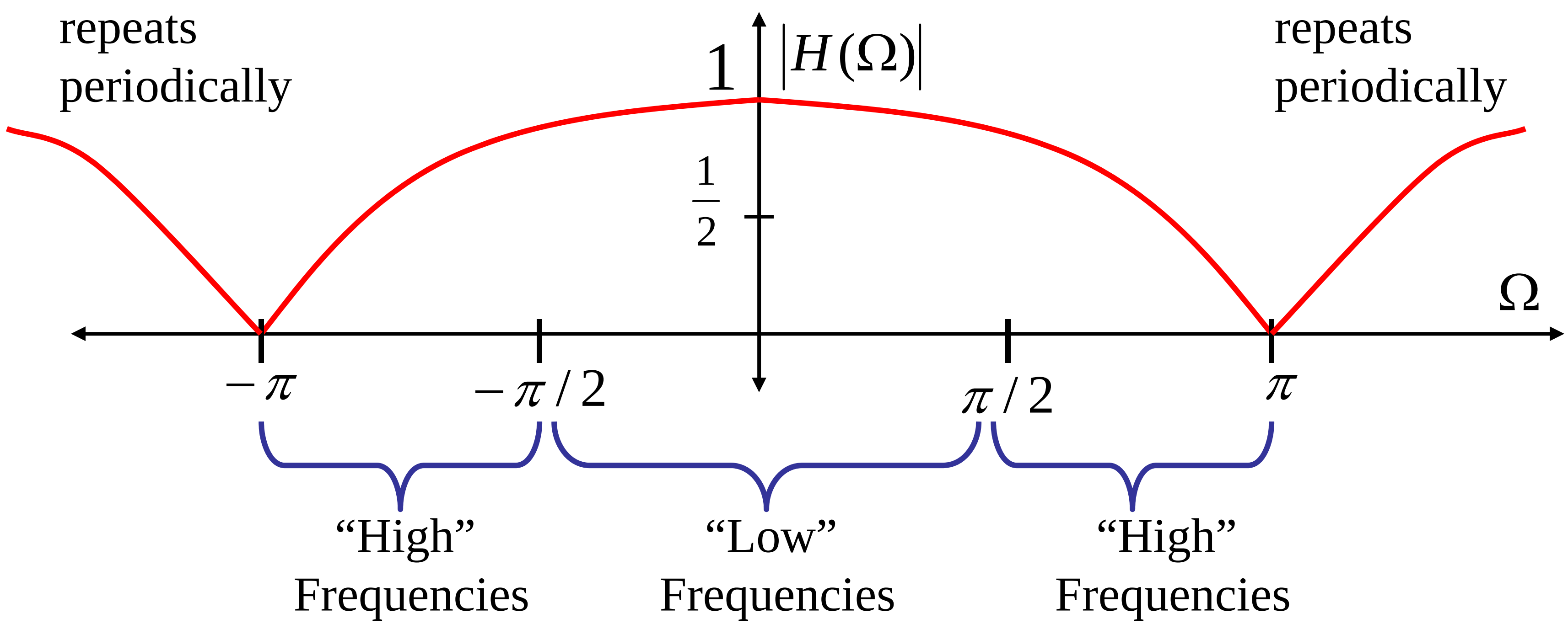
$$= \frac{1}{2}\sqrt{1 + 2\cos(\Omega) + \underbrace{\cos^2(\Omega) + \sin^2(\Omega)}_{=1}}$$

Trig. ID

$$= \frac{\sqrt{2}}{2}\sqrt{1+\cos(\Omega)} = \frac{1}{\sqrt{2}}\sqrt{1+\cos(\Omega)}$$

Now.. Plot this to see if it is a good LPF!

Here's a plot of this filter's freq. resp. magnitude:



Well…this does attenuate high frequencies but doesn't really "<u>stop</u>" them!

It is a low pass filter but not a very good one!

How do we make a better LPF???

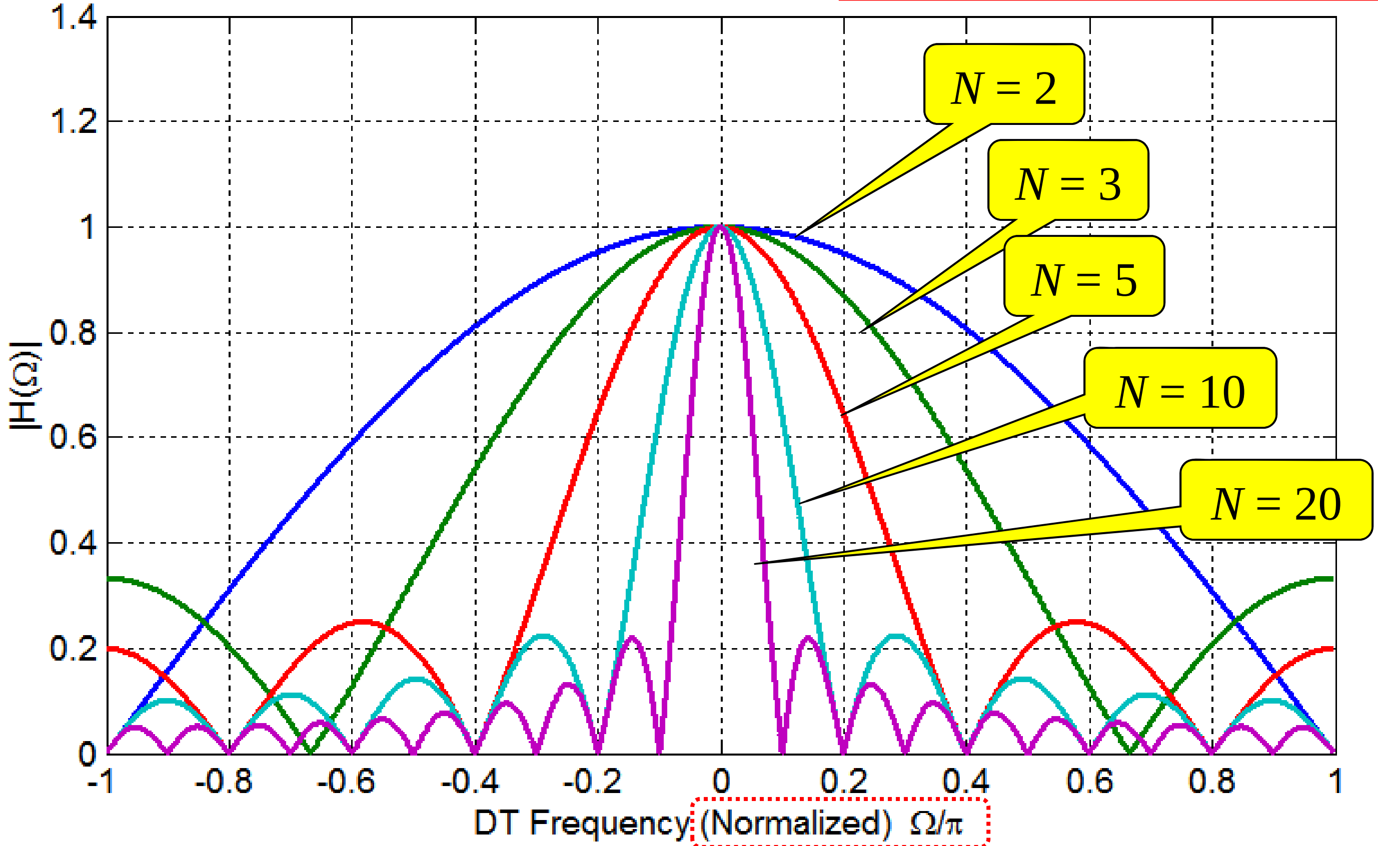
We could try "longer" non-recursive filters… having *N* terms:

$$\boxed{y[n] = \frac{1}{N}x[n] + \frac{1}{N}x[n-1] + \ldots + \frac{1}{N}x[n-(N-1)]}$$

Plots of frequency response for various N values…

$$y[n] = \frac{1}{N}x[n] + \frac{1}{N}x[n-1] + \ldots + \frac{1}{N}x[n-(N-1)]$$

$$H(\Omega) = \frac{1}{N} + \frac{1}{N}e^{-j\Omega} + \ldots + \frac{1}{N}e^{-j\Omega M}$$



Increasing the length causes the passband to get narrower… but the quality of the filter doesn't get better… so we generally need other types of filters.

We will see that better filters can be made from this form by allowing the "coefficients" to be non-uniform!

MATLAB Command to Compute DT Frequency Response.

H = freqz(b,a,w) **gives freq. resp. points in vector H at the frequency points in vector w.**

$$y[n]+a_1 y[n-1]+\ldots+a_N y[n-N]=b_0 x[n]+b_1 x[n-1]+\ldots+b_M x[n-M]$$

$$H(\Omega)=\frac{b_0+b_1 e^{-j\Omega}+\ldots+b_M e^{-j\Omega M}}{1+a_1 e^{-j\Omega}+\ldots+a_N e^{-j\Omega N}}$$

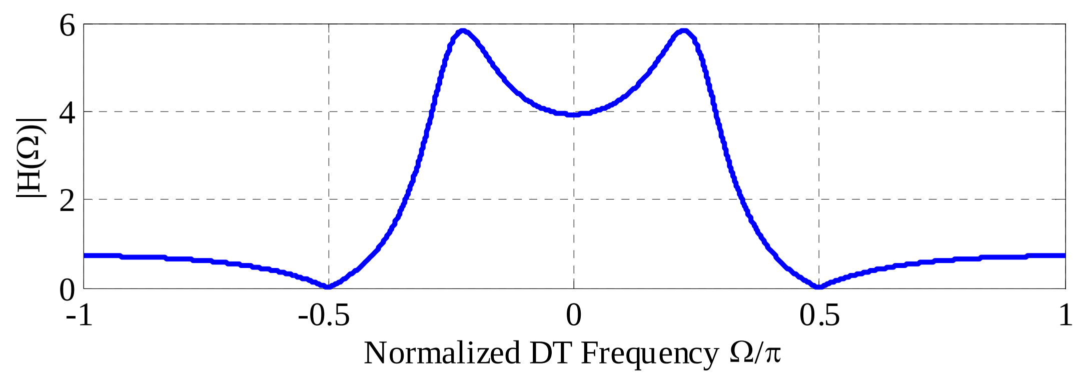
The numerator and denominator coefficients form the vectors b and a used in the freqz command.

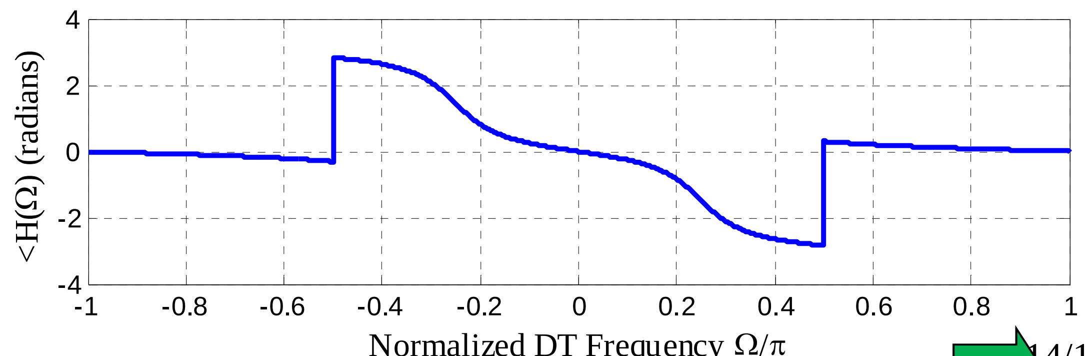
$$y[n]-1.1314\,y[n-1]+0.64\,y[n-2]=x[n]+x[n-2]$$

$$H(\Omega)=\frac{1+1e^{-j2\Omega}}{1-1.1314e^{-j\Omega}+0.64e^{-j2\Omega}}$$

```
>>  w=linspace(-pi,pi,2000);
>> a = [1 -1.1314 0.64];
>> b = [1 0 1];
>>  H=freqz(b,a,w);
>>  subplot(2,1,1)
>>  plot(w/pi,abs(H))
>>  subplot(2,1,2)
>>  plot(w/pi,angle(H))
```

Formatting commands are not shown here





Recall: Non-recursive filters have no “feedback”

$$\boxed{y[n] = b_0 x[n] + b_1 x[n-1] + \ldots + b_M x[n-M]}$$

$$\boxed{H(\Omega) = \frac{b_0 + b_1 e^{-j\Omega} + \ldots + b_M e^{-j\Omega M}}{1}} \Rightarrow \boxed{H(\Omega) = b_0 + b_1 e^{-j\Omega} + \ldots + b_M e^{-j\Omega M}}$$

```
>>  w=linspace(-pi,pi,2000);
>>  b = [1 2 1];
>>  H=freqz(b,1,w);
>>  subplot(2,1,1)
>>  plot(w/pi,abs(H))
>>  subplot(2,1,2)
>>  plot(w/pi,angle(H))
```

Formatting commands are not shown here