BINGHAMTON UNIVERSITY

State University of New York

EECE 301
Signals & Systems
Prof. Mark Fowler

Note Set #10

- C-T Signals: Circuits with Periodic Sources

Solving Circuits with Periodic Sources

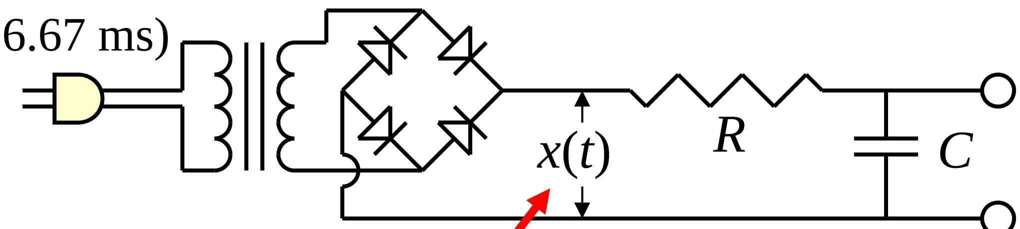
FS makes it easy to find the response of an RLC circuit to a periodic source!

- Use the FS to convert the source into a sum of sinusoids
- Do phasor analysis <u>for each</u> of the input sinusoids (think superposition!)
- Add up the sinusoidal responses to get the output signal

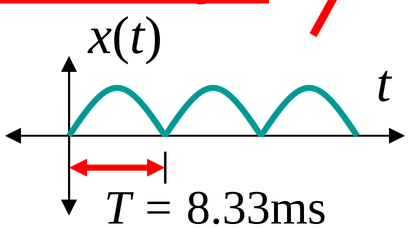
Obviously we can't do this for all infinitely many terms… but we can do it for enough… and if we do it numerically it is not hard!

<u>Example</u>: In electronics you have seen (or will see) how to use diodes and an RC filter circuit to create a DC power supply:

60Hz Sine wave
(Period = 16.67 ms)



<u>Periodic Signal</u>

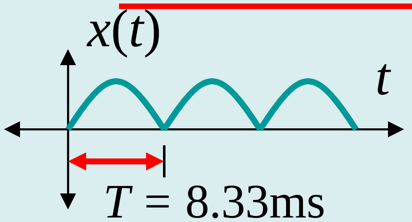


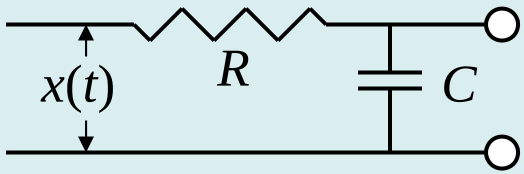
Periodic!! Think Fourier ***Series!!***

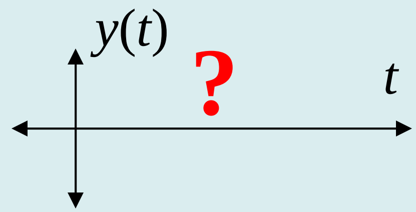
$$\omega_0 = 240\pi \text{ rad/sec}$$

Progression of Ideas

Periodic Source

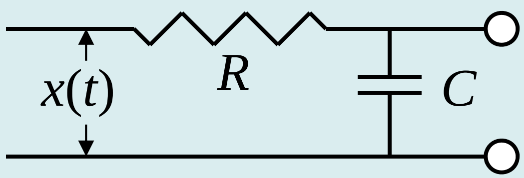


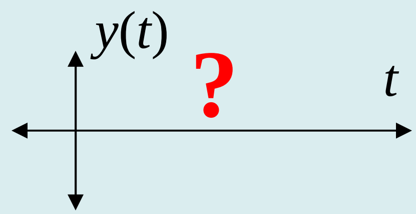




Periodic Source as FS

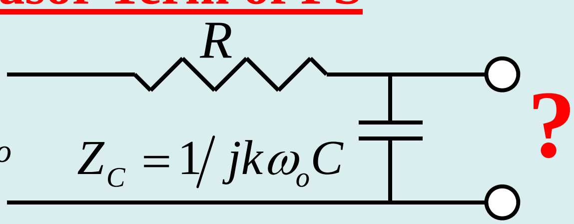
$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$





k^{th} Phasor Term of FS

c_k @ $k\omega_o$



$$d_k = \left[\frac{Z_C}{Z_C + R}\right] c_k = \left[\frac{1/jk\omega_o C}{1/jk\omega_o C + R}\right] c_k$$

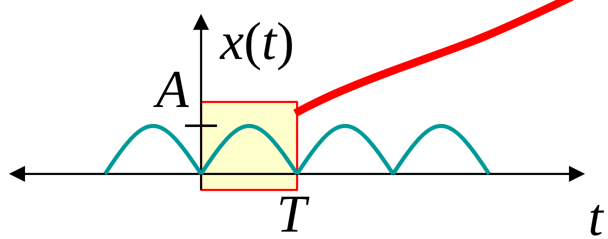
= Phasor to Find

$$= \left[\frac{1}{1 + jk\omega_o RC}\right] c_k$$

Need to find the c_k values… numerically or analytically

For this scenario we *can* find the c_k analytically…

The equation for the FS coefficients is: $c_k = \dfrac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt \qquad \omega_0 = \dfrac{2\pi}{T}$



Over this interval:

$$x(t) = A\sin\left(\frac{\pi}{T}t\right) \quad 0 \le t \le T$$

$$c_k = \frac{1}{T}\int_0^T A\sin\left(\frac{\pi}{T}t\right)e^{-jk\left(\frac{2\pi}{T}\right)t}dt$$

Now apply Calc I ideas to evaluate….

Change of variable: $\tau = \dfrac{\pi}{T}t$

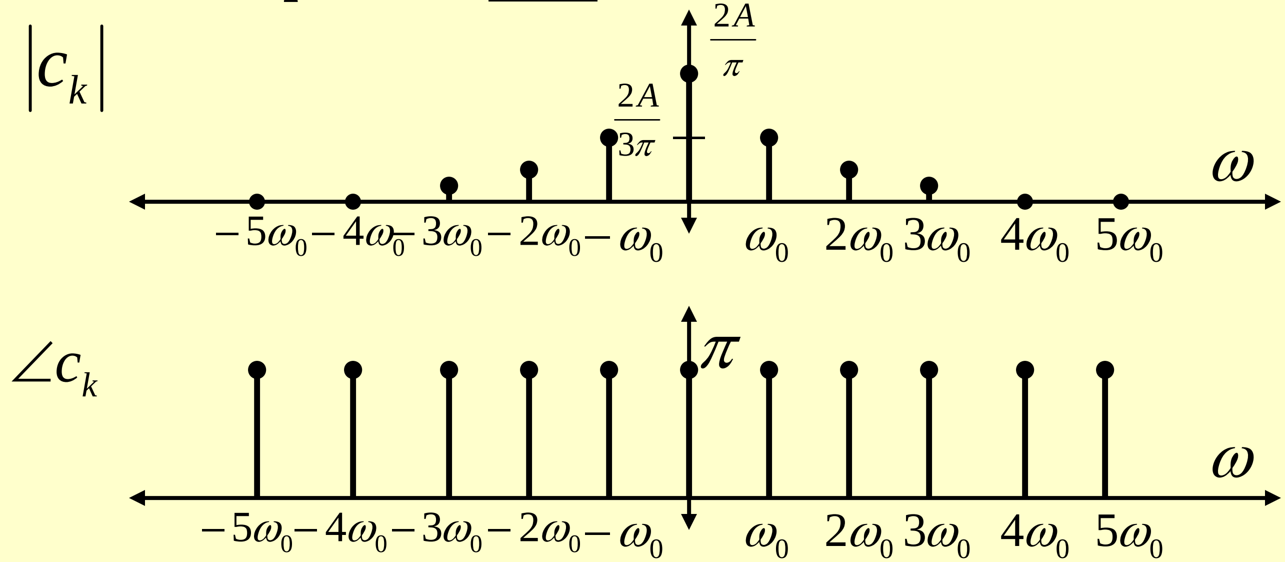
$$c_k = \frac{A}{\pi}\int_0^\pi \sin(\tau)e^{-jk2\tau}d\tau$$

Use a Table of Integrals and do some algebra & trig to get:

$$c_k = \frac{2A}{\pi(1-4k^2)}$$

FS coefficient for full-wave rectified sine wave of amplitude A

So the two-sided spectrum <u>after</u> the rectifier:



Now we can us Parseval's Theorem to determine how many terms we need in our approximation for the source…

$$P = \frac{1}{T}\int_0^T A^2 \sin^2\left(\frac{\pi}{T}t\right)dt = \frac{A^2}{\pi}\int_0^\pi \sin^2(\tau)\,d\tau = \frac{A^2}{\pi}\frac{\pi}{2} = \frac{A^2}{2}$$

$$P_{approx} = \sum_{k=-K}^{K}|c_k|^2 = \sum_{k=-K}^{K}\left|\frac{2A}{\pi(1-4k^2)}\right|^2 = \frac{4A^2}{\pi^2}\sum_{k=-K}^{K}\left|\frac{1}{(1-4k^2)}\right|^2$$

We can look at the ratio of these two as a good measure:

$$\frac{P_{approx}}{P} = \frac{\dfrac{4A^2}{\pi^2}\displaystyle\sum_{k=-K}^{K}\left|\frac{1}{(1-4k^2)}\right|^2}{\dfrac{A^2}{2}} = \frac{8}{\pi^2}\sum_{k=-K}^{K}\left|\frac{1}{(1-4k^2)}\right|^2$$

Numerically evaluating this for different K values shows that K = 10 retains more than 99.99% of the power. So we can use that value.

So… our numerical approach is now this:

1. Numerically evaluate c_k for k = –10 to 10
2. Numerically convert them into the d_k phasors
3. Convert the phasors into corresponding FS sinusoidal terms and add them up

$$c_k = \frac{2A}{\pi(1-4k^2)}$$

$$d_k = \left[\frac{1}{1+jk\omega_o RC}\right]c_k$$

$$y(t) \approx \sum_{k=-10}^{10} d_k e^{jk\omega_0 t}$$

We'll do this for:

- A = 10 volts
- R = 100 Ω
- C = 1000 μF

```
wo=240*pi;  % Set fund freq
fo=wo/(2*pi);  % convert to Hz
T = 2*pi/wo;   % compute period
K=10;   % Set number of terms
kv=(-K):K;  % set vector of k indices
A=10;  % set amplitude of input
R=100;  % set resistance
C=1000e-6;  % set capacitance

ck=(2*A/pi)./(1-4*(kv.^2));  % compute the input FS coefficents

dk=(1./(1+j*kv*wo*R*C)).*ck;  % compute the output FS coefficents

Fs = 4*K*fo;   % Compute sampling rate  (set here to twice the minimum value of 2Kfo)
Ts = 1/Fs;          % Compute sample spacing
t = (-3*T):Ts:(3*T);
x_apprx = zeros(size(t));   % sets up vector of zeros as first “partial sum”
for k = (-K):K     % loop through “all” coefficients
        x_apprx = x_apprx + ck(k+K+1)*exp(j*k*wo*t);    % Add current term to partial sum
end
x_apprx = real(x_apprx);
y_apprx = zeros(size(t));   % sets up vector of zeros as first “partial sum”
for k = (-K):K     % loop through “all” coefficients
        y_apprx = y_apprx + dk(k+K+1)*exp(j*k*wo*t);    % Add current term to partial sum
end
y_apprx = real(y_apprx);    % theory says imaginary parts cancel… so enforce this in case
           % of numerical round-off issues

figure(1); plot(t,x_apprx,'r',t,y_apprx,'g--'); xlabel('time (seconds)'); ylabel('Input and Output (volts)'); grid
figure(2); subplot(2,1,1); stem(kv,abs(ck)); subplot(2,1,2); stem(kv,abs(dk))
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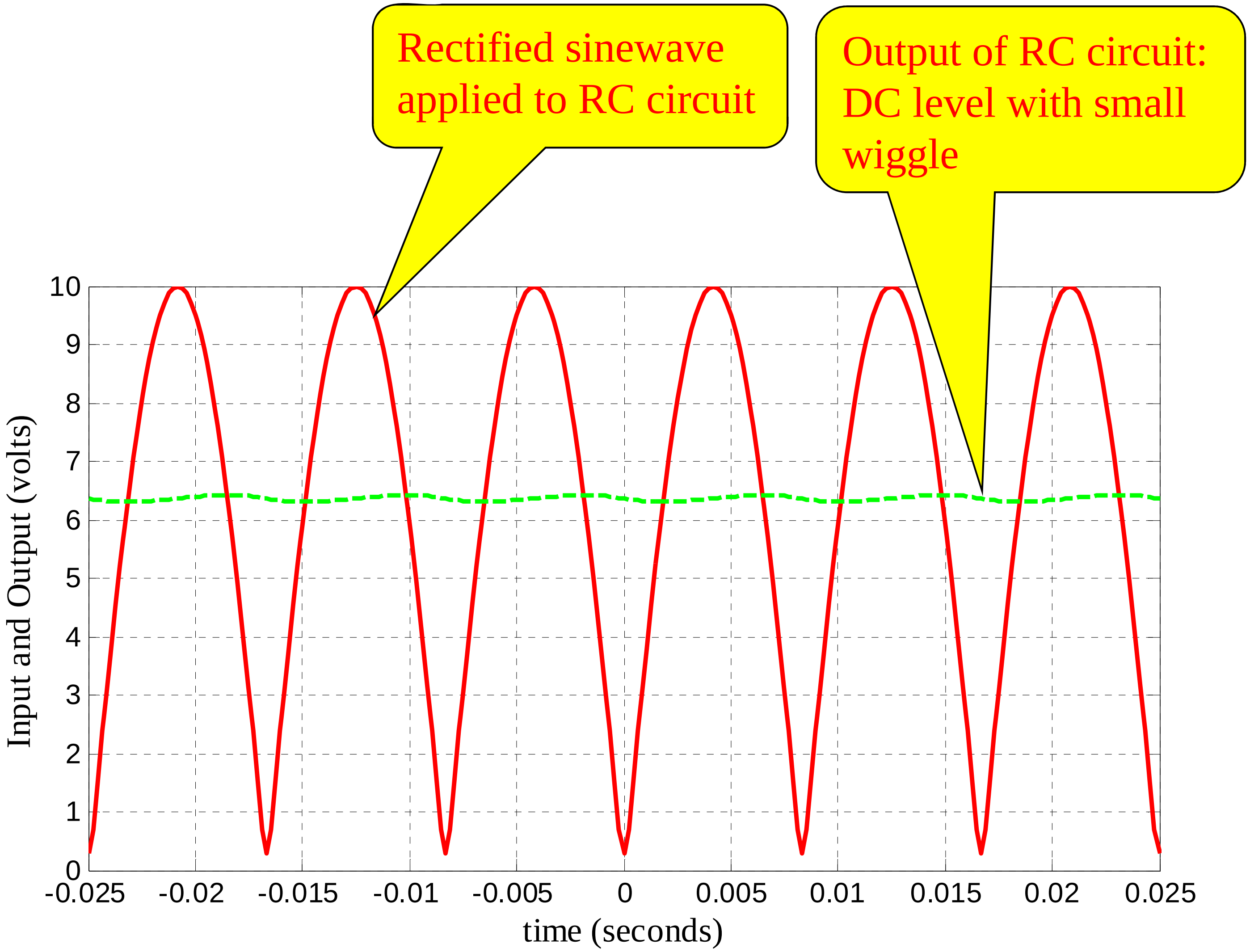
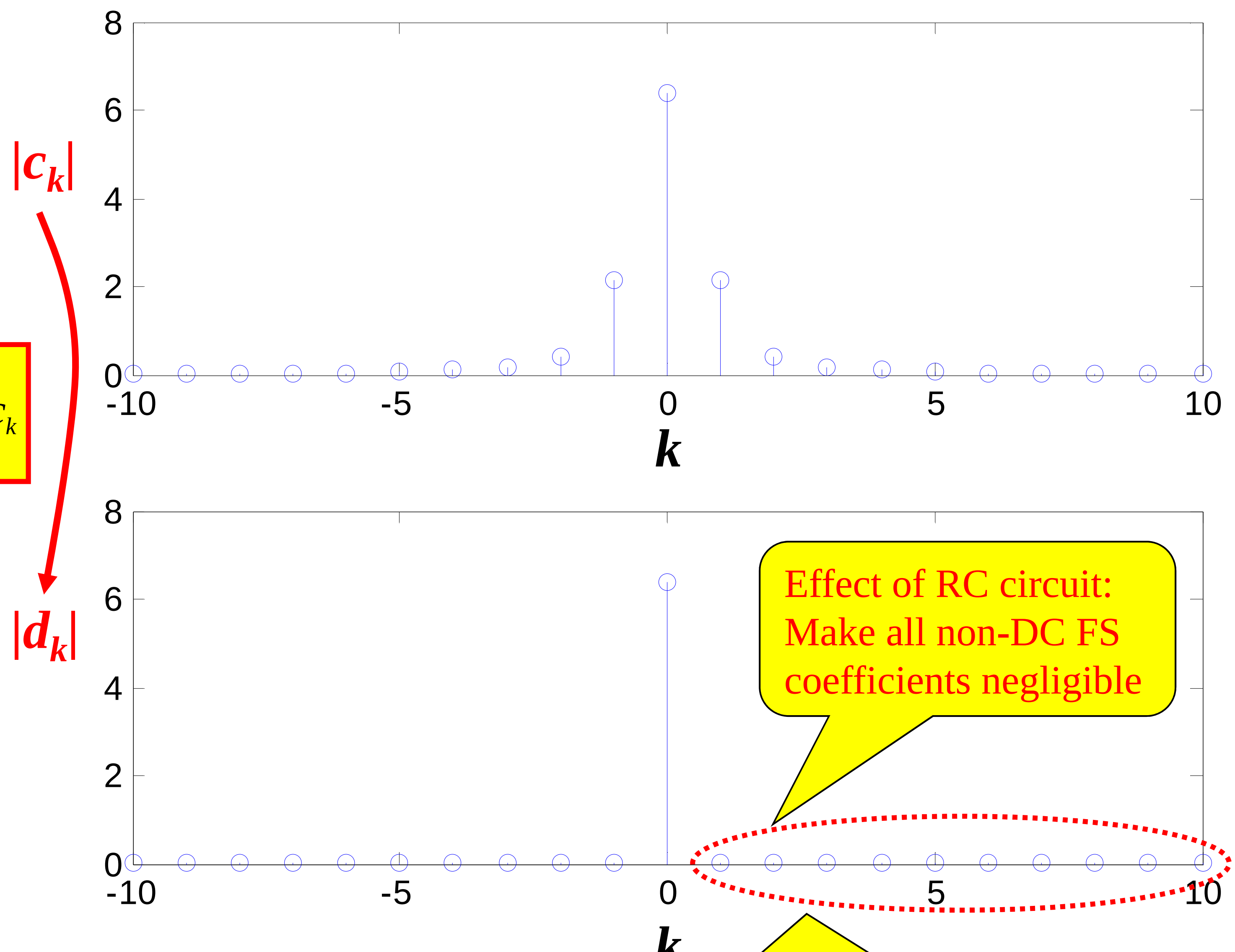


Rectified sinewave applied to RC circuit
Output of RC circuit: DC level with small wiggle
10
9
8
7
6
5
4
3
2
1
0
-0.025
-0.02
-0.015
-0.01
-0.005
0
0.005
0.01
0.015
0.02
0.025
Input and Output (volts)
time (seconds)

$$d_k = \left[\frac{1}{1 + jk\omega_o RC} \right] c_k$$

Changed via Multiplication!



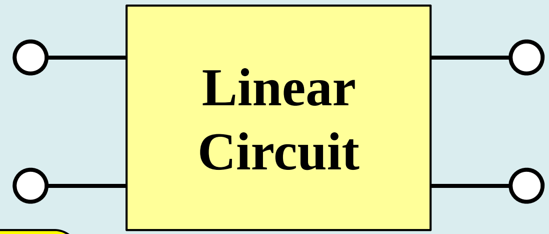
Multiplicative factor has small magnitude here!

Big Idea: "Frequency Response"

Input

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Input's FS Coefficients



Output

$$y(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_0 t}$$

Output's FS Coefficients

$$d_k = H(k\omega_o)c_k$$

$H(\omega)$ is the "Frequency Response" of the Circuit

How to find the Frequency Response of a Circuit…

- Assume arbitrary phasor X with frequency ω
- Analyze circuit to find output phasor Y
  - It will always take this multiplicative form: $Y = H(\omega)\, X$
  - All impedances are evaluated at the arbitrary frequency ω
- The frequency response function $H(\omega)$ is the thing that multiplies X