BINGHAMTON UNIVERSITY
State University of New York

EECE 301
Signals & Systems
Prof. Mark Fowler

Note Set #9

- C-T Signals: FS Spectrum

Trig Form "Spectrum"… Is "Single Sided"

Best for "thinking about real-world ideas"

Trig Form: Amplitude & Phase

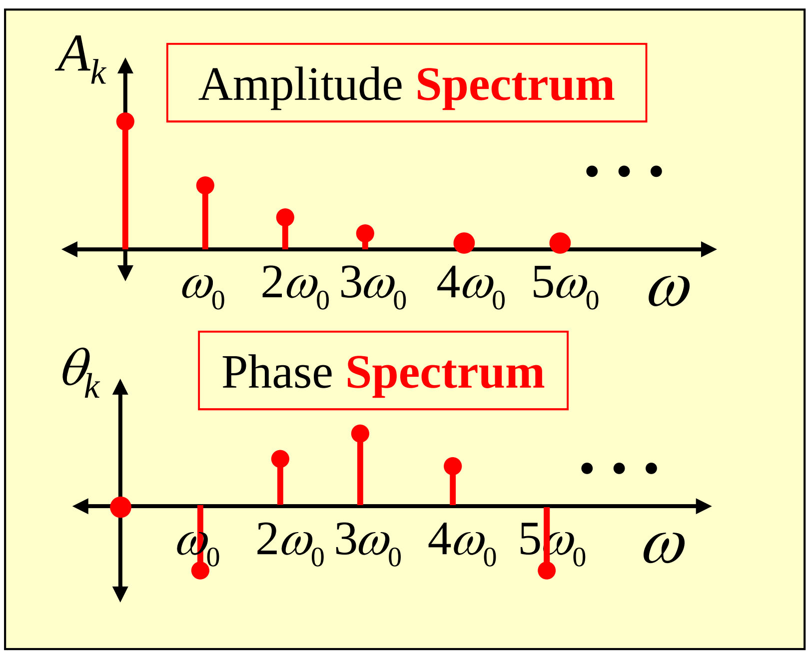
$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$$

Need A_k and θ_k
for $k = 0, 1, 2\ldots$

A_k = Amplitude
θ_k = Phase

So… to describe a signal via FS we specify:
"Amplitude & Phase @ Each Frequency"

A good way to "see" the FS coefficients is by plotting them vs. frequency:



For this form of FS:

- Do not need negative freqs
  ➔ "Single Sided" Spectrum

Exp Form "Spectrum"… Is "Double Sided"

Best for "doing math" (c_k are like phasors!!)

Exponential Form

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

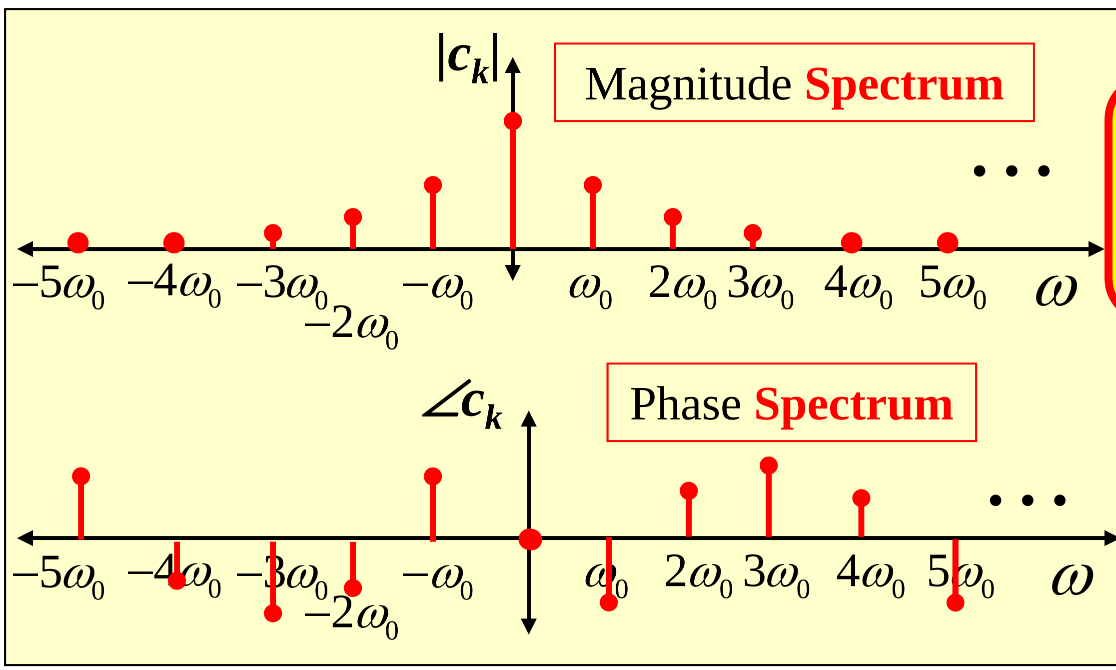
Need c_k (complex!) for $k = \ldots -2, -1, 0, 1, 2\ldots$

$|c_k|$ = Magnitude
$\angle c_k$ = Phase

$$c_k e^{jk\omega_0 t} = \left[|c_k| e^{j\angle c_k} \right] e^{jk\omega_0 t}$$

$$= |c_k| e^{j(k\omega_0 t + \angle c_k)}$$

So… to describe a signal via FS we specify:
"Magnitude & Phase @ Each Frequency"



For this form of FS:

- Do need negative freqs
  - ➔ "Double Sided" Spectrum

Spectrum Characteristics

Trig Form: Amplitude & Phase

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$$

Exponential Form

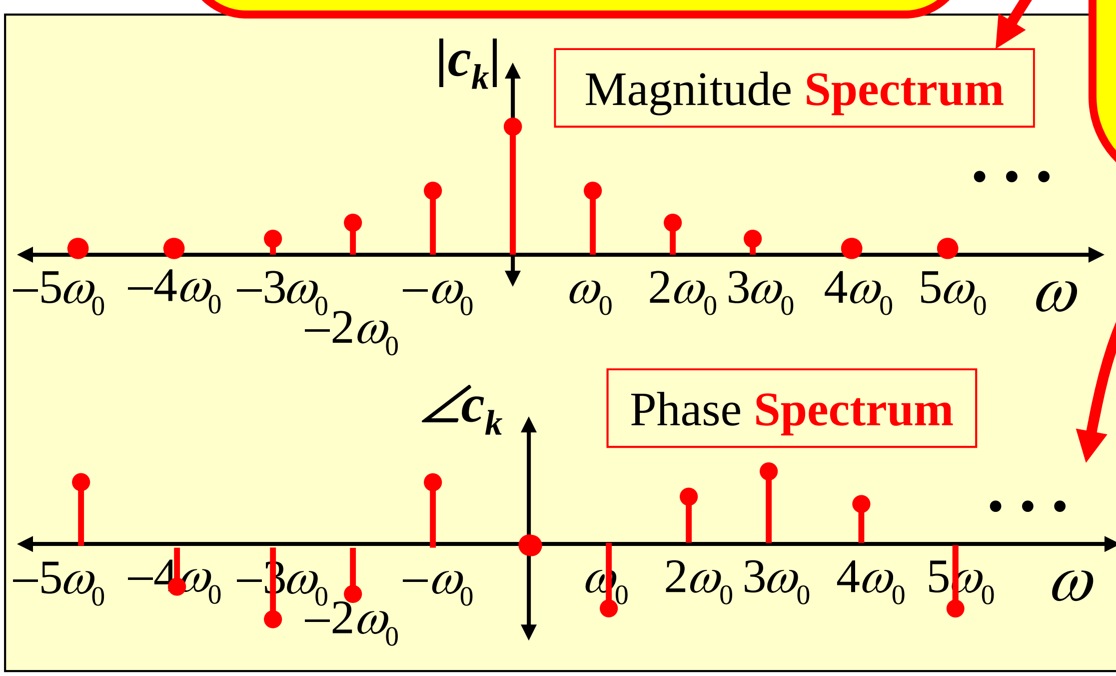
$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

For Trig Form of FS Spectrum:

- "Single Sided" Spectrum
- $A_k \geq 0$ for $k > 0$
  - A_0: positive or negative
- θ_k is in radians $\theta_0 = 0$

For Exp Form of FS Spectrum:

- "Double Sided" Spectrum
- $|c_k| \geq 0$ for all k
  - *Even Symmetry for Magn.*
- $\angle c_k$ is in radians
- $\angle c_0 = 0$ or $\pm\pi$
- $\angle c_k = -\angle c_{-k}$
  - *Odd Symmetry for Phase*



$$\left.\begin{aligned} c_k &= \tfrac{1}{2} A_k e^{j\theta_k} \\ c_{-k} &= \tfrac{1}{2} A_k e^{-j\theta_k} \end{aligned}\right\} k = 1, 2, 3, \ldots$$

Parseval's Theorem

We saw earlier how to compute the average power of a periodic signal if we are given its time-domain model:

$$P = \frac{1}{T}\int_{t_0}^{T+t_0} x^2(t)\,dt$$

Q: Can we compute the average power from the frequency domain model

$$\{c_k\}, \quad k = 0, \pm 1, \pm 2, \ldots$$

A: Parseval's Theorem says… Yes!

Parseval's theorem says that the avg. power can be computed this way:

$$P = \sum_{k=-\infty}^{\infty} |c_k|^2$$

c_k are the Exp. Form FS coefficients

$$\frac{1}{T}\int_{t_0}^{t_0+T} x^2(t)\,dt = \sum_{k=-\infty}^{\infty} |c_k|^2$$

Left side is clearly finite for real-world signals…

Thus, the $|c_k|$ must decay fast enough as $k \to \pm\infty$

Tells us something about how the magnitude spectrum should look!

Interpreting Parseval's Theorem

$$\frac{1}{T}\int_{t_0}^{t_0+T} x^2(t)dt = \sum_{k=-\infty}^{\infty} |c_k|^2$$

"sum" of squares in time-domain model = **"sum" of squares in freq.-domain model**

$x^2(t)$ = power at time t (includes effects of all frequencies)

We can find the power in the time domain by "adding up" all the "powers at each time"

$|c_k|^2$ = power at frequency $k\omega_0$ (includes effects of all times)

We can find the power in the frequency domain by adding up all the "powers at each frequency"

One Use for Parseval's Theorem

When numerically computing the FS approximation… PT allows you to compute the power of the error term:

First find Avg Power: $$P = \frac{1}{T}\int_{t_0}^{T+t_0} x^2(t)dt$$ Do analytically or numerically

Then find power of approximate using PT: $$P_{approx} = \sum_{k=-K}^{K} |c_k|^2 = |c_0|^2 + 2\sum_{k=1}^{K}|c_k|^2$$

Then find power of error as $P_{error} = P - P_{approx}$

It is easy to show that $$P_{error} = 2\sum_{k=K+1}^{\infty}|c_k|^2$$

Since the $|c_k|$ decay as $\boldsymbol{k} \to \infty$ this shows that we can make P_{error} as small as we want by making K big enough!