BINGHAMTON UNIVERSITY
State University of New York

EEO 401
Digital Signal Processing
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Note Set #9

- Using ZT to Analyze DT LTI Systems
- Reading Assignment: Sect. 3.5 of Proakis & Manolakis

Transient and Steady-State Responses

We can use the ZT to get an idea of what to expect the output of an LTI system will look like.

$$\underbrace{y[n]+\sum_{i=1}^{N} a_i y[n-i]}=\underbrace{\sum_{i=0}^{M} b_i x[n-i]}$$

Contains $x[n]$, $x[n\text{-}1]$,…

This system is causal, we won't have $x[n+1]$, $x[n+2]$, etc. here

$$A^z(z) = z^N + a_1 z^{N-1} + \ldots + a_{N-1} z + a_N$$

$$B^z(z) = b_0 z^N + b_1 z^{N-1} + \ldots + b_M z^{N-M}$$

$$\boxed{H^z(z) = \frac{B^z(z)}{A^z(z)}}$$

Assuming zero ICs and using the convolution property:

$$Y^z(z) = \underbrace{H^z(z)}_{H(z)\,=\,\text{transfer function}} X^z(z) = \frac{B^z(z)}{A^z(z)} X^z(z)$$

General result: Get output's ZT by multiplying TF by input's ZT. But… we can study this further to get key insight!!

For simplicity assume $X^z(z) = E^z(z)/F^z(z)$

For simplicity… assume <u>distinct</u> poles

$$Y^z(z) = \frac{B^z(z)}{A^z(z)}\frac{E^z(z)}{F^z(z)} = \frac{c_1 z}{z-p_1} + \frac{c_2 z}{z-p_2} + \cdots + \frac{c_N z}{z-p_N} + \frac{D^z(z)}{F^z(z)}$$

Some Polynomial that "falls out of" PFE

$$y_{zs}[n] = \underbrace{c_1 p_1^n u[n] + c_2 p_2^n u[n] + \cdots + c_N p_N^n u[n]}_{\textbf{Transient Response…Decays if } |p_i| < 1} + y_{ss}[n]$$

Steady State Response

So… if all the poles are inside the UC then the Transient response decays and is not that interesting for most signal processing applications.

Causality & ROC

Recall that a causal system with 0 ICs can not give a non-zero output until the input becomes non-zero.

Since $h[n]$ is the output due to $\delta[n]$ with zero ICs we can see that a causal system must have

$$h[n] = 0 \ \forall n < 0$$

We also know that for a causal sequence the ROC of its ZT is the exterior of a circle. Thus the ROC of $H^z(z)$ for a causal system is the exterior of a circle.

An LTI System is causal if an only if the ROC of its transfer function $H^z(z)$ is the exterior of a circle of radius $R < \infty$, including the point $z = \infty$.

<u>Stability & ROC</u>

We've already discussed that a necessary and sufficient condition for an LTI system to be stable is for the impulse response to be absolutely summable:

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

From this we can derive insight into the impact of the ROC on stability. Start from definition of $H^z(z)$:

$$H^z(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n} \quad \Rightarrow \quad |H^z(z)| \le \sum_{n=-\infty}^{\infty} \left|h[n]z^{-n}\right| = \sum_{n=-\infty}^{\infty} |h[n]|\left|z^{-n}\right|$$

Now evaluating this inequality on the unit circle shows

$$\left|H^z(z)\right|_{|z|=1} \le \sum_{n=-\infty}^{\infty} |h[n]| \quad \Rightarrow \quad \left|H^z(z)\right|_{|z|=1} \le \infty \quad \text{for a stable system}$$

Thus… If the system is BIBO stable then the UC is in the ROC.
Can prove the reverse is true as well.

An LTI System is <u>BIBO Stable</u> if an only if the UC is in the ROC.

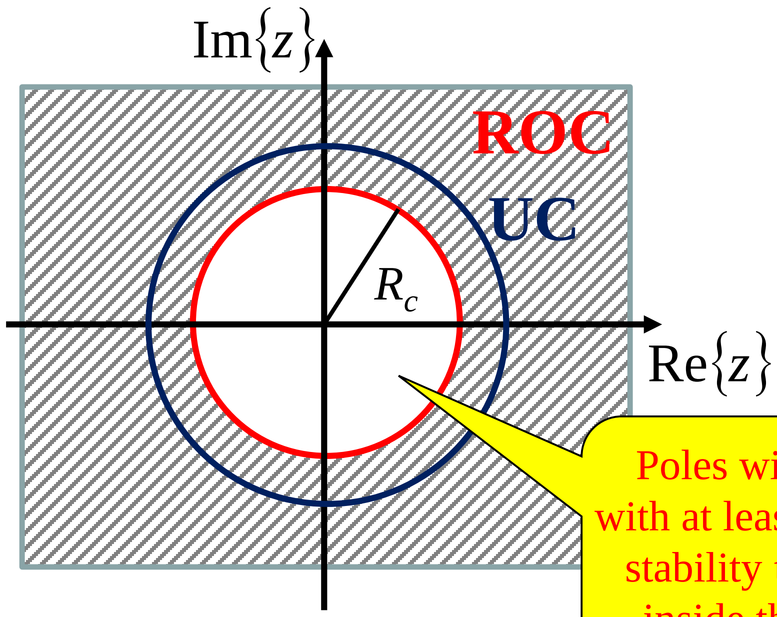
For a causal system… we now the ROC is the outside of a circle.

Thus, that circle must be inside the Unit Circle…

And… since the poles of a stable system must be outside the ROC…

An LTI Causal System is BIBO Stable if an only if all the poles are strictly inside the UC.

Causal System ROC



Poles will be inside this circle… with at least 1 pole on the circle. For stability the poles must be strictly inside the UC… not on the UC.

Stability & Poles on the Unit Circle

On the previous slide we said explicitly that the poles must be strictly inside the UC for stability. To show that we can't have a pole on the unit circle and still have BIBO all we need is one counter example.

Counter Example: Consider a system with a single pole $H(z) = \dfrac{z}{z-a}$

Consider an input with ZT $X(z) = \dfrac{z}{z-b}$

For simplicity let a,b be real and positive   But let $0 < b \le 1$ so the input is bounded

Then the output ZT is $Y(z) = H(z)X(z) = \left[\dfrac{z}{z-a}\right]\left[\dfrac{z}{z-b}\right]$

For $a \neq b$ we find that the IZT of this will have the form

$$y[n] = Aa^n u[n] + Bb^n u[n]$$

Remains bounded even if pole is ***on*** the UC

For $a = b$ we find that the IZT of this will have the form

$$y[n] = Aa^n u[n] + Bna^n u[n]$$

Bounded only if pole is strictly ***inside*** the UC

Unbounded if pole is ***on*** the UC

Exercise: Consider the case where there are two poles @ $z = 1$

Summary of Stability Results

A causal system is

- …BIBO ***stable*** if all poles lie strictly inside the UC
- …BIBO ***marginally stable*** if there are some single poles on the UC but no poles outside the UC
- …BIBO ***unstable*** if there is at least one pole outside the UC and/or at least one multiple pole on the UC

Pole – Zero Cancellation

Sometimes when finding $Y^z(z)$ the combination of z-transforms leads to a pole and a zero at the same location in the plane.

This can happen either from:

- the interaction of $H^z(z)$ and $X^z(z)$
- when cascading multiple systems together
  - which we saw leads to a composite transfer function that is the product of the cascaded transfer function

Example: Cascade of two systems $H_1^z(z) = \dfrac{z(2z+1)}{(z-0.3)(z-2)}$ $\quad H_2^z(z) = \dfrac{(z-2)}{(z+0.5)}$

First system is NOT stable!!!

The total transfer function is

$$H^z(z) = H_1^z(z)H_2^z(z) = \left[\frac{z(2z+1)}{(z-0.3)\cancel{(z-2)}}\right]\left[\frac{\cancel{(z-2)}}{(z+0.5)}\right] = \frac{z(2z+1)}{(z-0.3)(z+0.5)}$$

- In theory this kind of "stabilization" by pole cancellation works…
- But… in practice, numerical precision issues in the implementation may result in imperfect cancellation… so beware of this!

Example: Similarly we can have pole-zero cancellation between the system and the input

$$y[n] - \tfrac{5}{6} y[n-1] + \tfrac{1}{6} y[n-2] = x[n]$$

$$x[n] = \delta[n] - \tfrac{1}{3}\delta[n-1]$$

The transfer function is: $H^z(z) = \dfrac{1}{1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}} = \dfrac{1}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{3}z^{-1}\right)}$

The ZT of the input is: $X^z(z) = 1 - \tfrac{1}{3}z^{-1}$

So… after cancellation the output ZT is $Y^z(z) = \dfrac{1}{1 - \frac{1}{2}z^{-1}}$

And.. then the time-domain form for the output signal is

$$y[n] = \left(\tfrac{1}{2}\right)^n u[n]$$

Note… from our earlier studies we would expect the output to include an exponential term for each pole in its transient response. But here we only see one of the pole's exponential…

This input only "excites" one of the two poles!

Stability of 2nd-Order Systems

As we'll see later… 2-pole sub-systems form the building blocks for realizing higher-order systems. Thus, having explicit insight into the stability of 2nd order systems is very useful.

2nd Order Diff. Eq.: $$y[n] = -a_1 y[n-1] - a_2 y[n-2] + b_0 x[n]$$

2nd Order Transfer Function: $$H(z) = \frac{b_0}{1 + a_1 z^{-1} + a_2 z^{-2}} = \frac{b_0 z^2}{\boxed{z^2 + a_1 z + a_2}}$$

2nd Order Poles: $$p_1, p_2 = \frac{-a_1}{2} \pm \sqrt{\frac{a_1^2 - 4a_2}{4}}$$

$$\Rightarrow \quad a_1 = -\left(p_1 + p_2\right), \quad a_2 = p_1 p_2$$

$$\Rightarrow \quad |a_2| = |p_1 p_2| = |p_1||p_2| < 1, \quad |a_1| < 1 + a_2$$

These provide the "Stability Triangle"

$$a_1^2 - 4a_2 = 0 \quad \Rightarrow \quad a_2 = \frac{a_1^2}{4}$$

… provides the dividing curve between real roots and complex roots

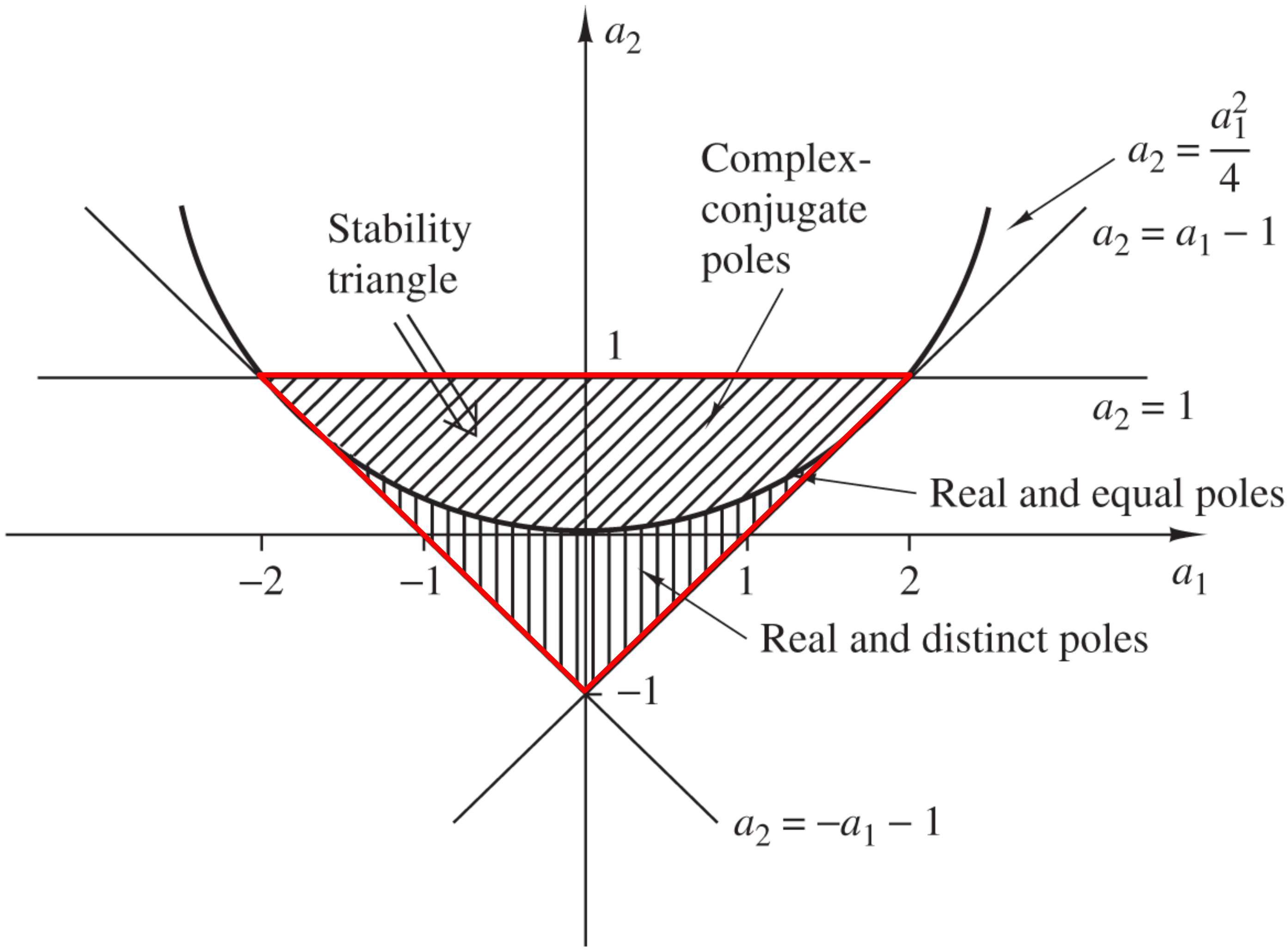


Figure 3.5.1 Region of stability (stability triangle) in the (a_1, a_2) coefficient plane for a second-order system.