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UNIVERSITY
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EEO 401
Digital Signal Processing

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Note Set #8

- Rational Z-Transforms… Diff Equations… Transfer Function
- Reading Assignment: Sect. 3.3 of Proakis & Manolakis

In Section 3.3 of the textbook they state that there are important cases when the ZT is a rational function of z. At first they use a generic *signal* x[n] and its ZT $X^z(z)$… However, the most likely time you'll see a ZT that is a rational function is when talking about the transfer function $H^z(z)$ of an LTI system I will stress that here right from the beginning.

First we'll see how/why we get a transfer function that is a rational function of z… then we will see what impact that has on the system behavior and how wew can use that insight to understand how system's will behave…

Finding the Transfer Function from Difference Eq.

The concept of Transfer Function arises in the context of zero initial conditions. So… imagine that we have a Diff. Eq. with zeros ICs and we take its ZT using the time shift property…

$$y[n] + a_1 y[n-1] + \ldots + a_N y[n-N] = b_0 x[n] + b_1 x[n-1] + \ldots + b_M x[n-M]$$

ZT

$$ZT\left\{y[n] + a_1 y[n-1] + \ldots + a_N y[n-N]\right\} = ZT\left\{b_0 x[n] + b_1 x[n-1] + \ldots + b_M x[n-M]\right\}$$

Delay Prop. & Algebra

$$Y(z)\left[1 + a_1 z^{-1} + \ldots + a_N z^{-N}\right] = X(z)\left[b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}\right]$$

Algebra

$$Y(z) = \underbrace{\left[\frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{1 + a_1 z^{-1} + \ldots + a_N z^{-N}}\right]}_{H(z)} X(z)$$

So… can just write H(z) by inspection of D.E. coefficients!

More generally there can be an a_0 but we can always normalize it to 1

Poles and Zeros of Transfer Function

$$H(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{1 + a_1 z^{-1} + \ldots + a_N z^{-N}} \;\Rightarrow\; H(z) = z^{(N-M)} \frac{b_0 z^M + b_1 z^{M-1} + \ldots + b_M}{z^N + a_1 z^{N-1} + \ldots + a_N}$$

Define the ***polynomials*** $A(z)$ and $B(z)$ so that:

$$H(z) = z^{(N-M)} \frac{B(z)}{A(z)}$$

Assume there are no common roots in the numerator $B(z)$ and denominator $A(z)$.
(If not, assume they've been cancelled and redefine $B(z)$ and $A(z)$ accordingly)

Poles of *H(z)*: The values on the complex z-plane where $|H(z)| \to \infty$

Zeros of *H(z)*: The values on the complex z-plane where $|H(z)| = 0$

The roots of the **denominator** polynomial $A(z)$ determine N poles.

The roots of the **Numerator** polynomial $B(z)$ determine M zeros.

The term $z^{(N-M)}$ gives poles/zeros at the origin according to:

- If $N > M$: $\boldsymbol{N - M}$ **zeros @ Origin**
- If $N < M$: $\boldsymbol{M - N}$ **poles @ Origin**

Plotting the locations of the poles and zeros on the z-plane is called the **Pole-Zero plot** of the TF

Example: Finding Poles and Zeros

$$y[n]-\tfrac{1}{\sqrt{2}}y[n-1]+\tfrac{1}{4}y[n-2]=2x[n]+x[n-1]$$

$$H(z)=\frac{2+z^{-1}}{1-\frac{1}{\sqrt{2}}z^{-1}+\frac{1}{4}z^{-2}}$$

$$H(z)=\frac{z(2z+1)}{z^{2}-\frac{1}{\sqrt{2}}z+\frac{1}{4}}$$

zeros: $z=0,\ z=-1/2$

poles: $z=\frac{1}{2\sqrt{2}}(1\pm j)$

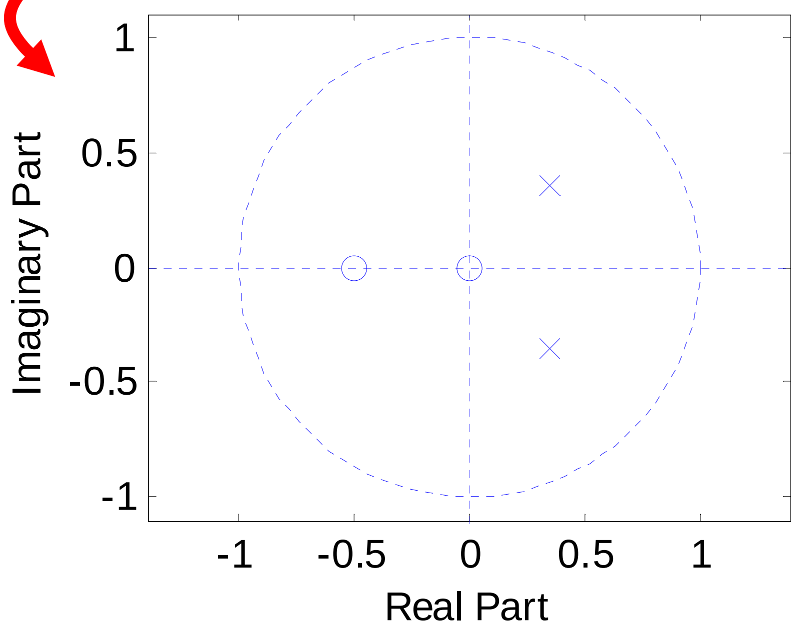
p =1 zero at origin

Conjugate Pair

Using MATLAB too make a Pole-Zero Plot

```
>> zplane([2 1],[1 -(1/sqrt(2)) 1/4])
```

Coeff. Vectors ***MUST*** be **<u>rows</u>**



Pole-Zero Plot
x marks poles
o marks zeros
(Use a number next to the symbol to indicate "repeated" roots)

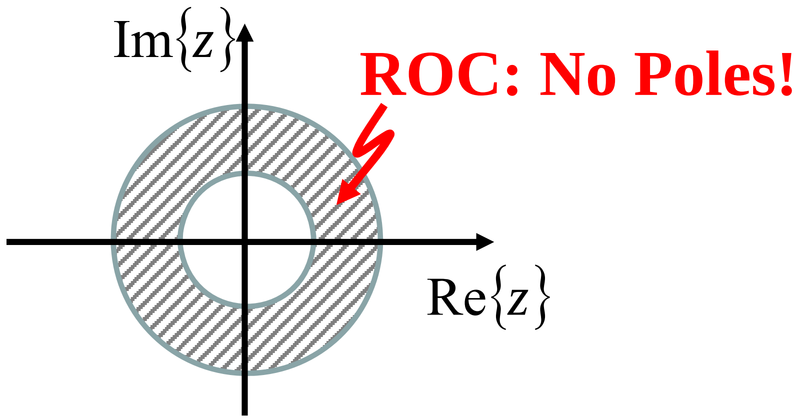
Poles and the ROC

By its very definition, the transfer function magnitude goes to ∞ at a pole:

Let $z = p$ be a pole of $H^z(z)$. Then $\lim_{z \to p} \left| H^z(z) \right| = \infty$

That means that the summation in the ZT definition does not converge at z values that are poles.

Thus… the ROC must be devoid of poles!!



In fact… there will be poles right on the edge of the ROC… If there were not then the ROC can be extended without hitting a z-point where $H^z(z) \to \infty$!!!

Example: Poles and ROC

Changed from previous example

$$y[n]-\tfrac{1}{\sqrt{2}}\,y[n-1]+\tfrac{1}{4}\,y[n-2]=x[n]+x[n-1]$$

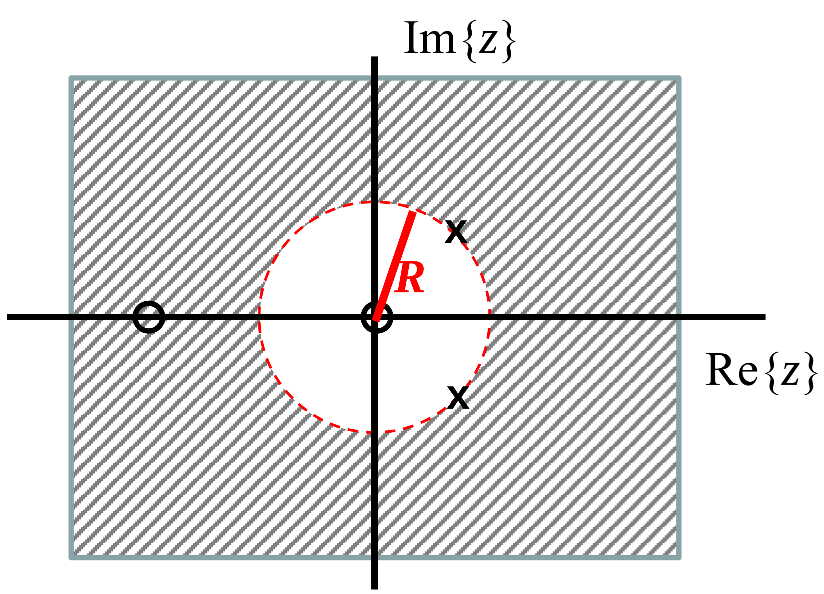
We already found the poles (and zeros) for this system:

zeros: $z=0,\ z=-1$

poles: $z=\dfrac{1}{2\sqrt{2}}(1\pm j)$

Clearly… this is a causal system: put in $\delta[n]$ (with zero ICs) and recursively solve for $h[n]$ and you'll see that it is causal.

So… we know that the ROC will be outside a circle. And… that the edge of the ROC will be at the poles



To find the radius of the ROC edge's circle… find the magnitude of the poles:

$$R=\left|\frac{1}{2\sqrt{2}}(1\pm j)\right|=\frac{1}{2\sqrt{2}}\left|(1\pm j)\right|$$

$$=\frac{1}{2\sqrt{2}}\sqrt{1^2+1^2}=\frac{1}{2}$$

Relationship: Transfer Function and Freq. Resp.

Recall: DTFT = ZT evaluated on Unit Circle… if UC is inside ROC

Fact: For causal systems UC is inside ROC if all poles are inside UC

$$H(\Omega) = H(z)\Big|_{z=e^{j\Omega}}$$

If all poles are inside the UC

We saw how to use freqz before to plot the Frequency Response… this just shows how to plot the Frequency Response from the Transfer Function coefficients:

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-N}}$$

\>> num = $[b_0 \; b_1 \ldots b_M]$ ← must put any zero b_i into the vector

\>> den = $[a_0 \; a_1 \ldots a_N]$ ← must put any zero a_i into the vector

\>> omega = -pi: ?:pi (Pick appropriate spacing)

\>> H = freqz(num, denom, omega)

\>> plot(omega/pi, abs(H))

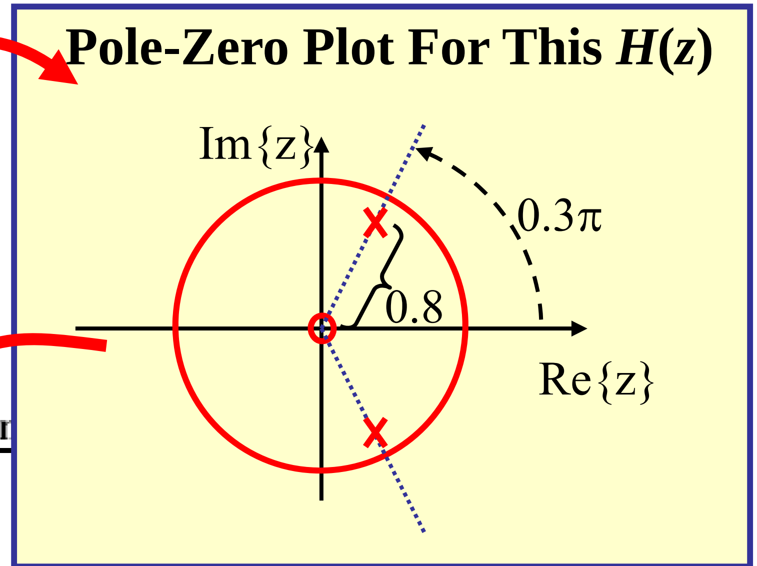
\>> plot(omega/pi, angle(H))

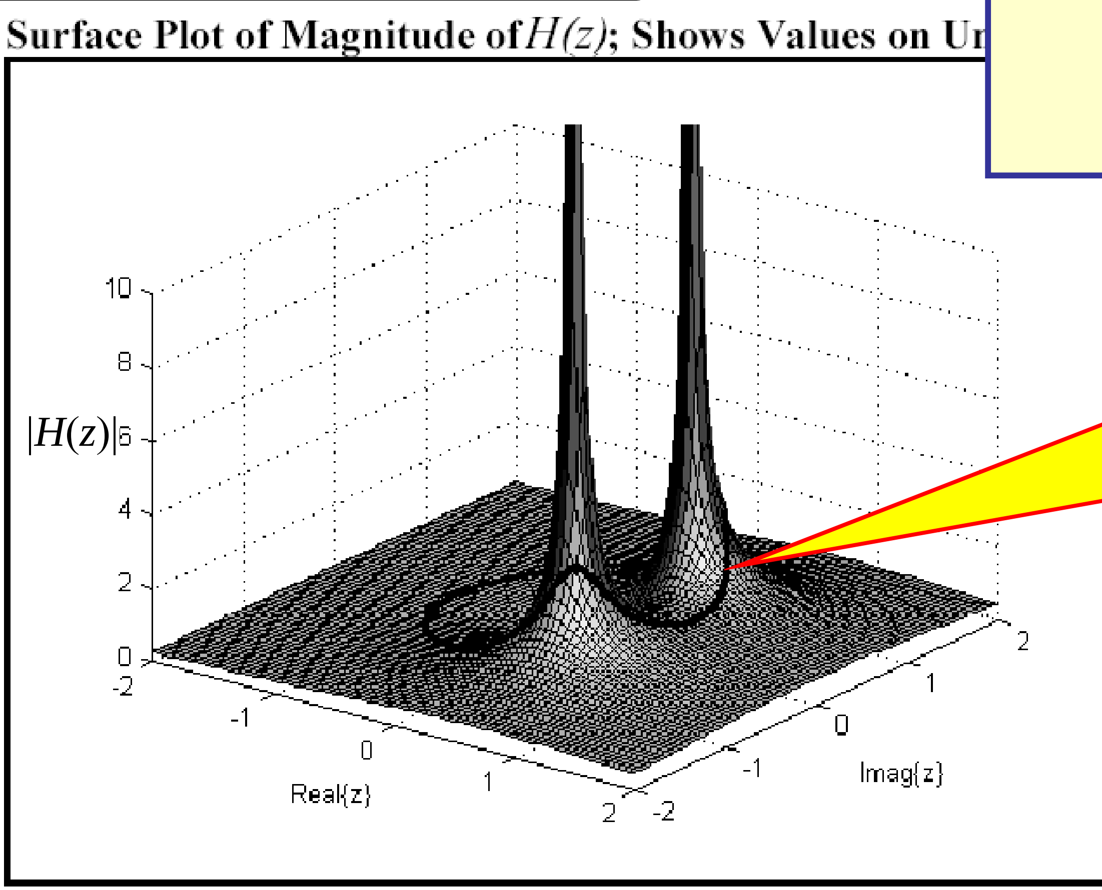
Visualizing Relationship Between TF & FR

Zero @ z = 0

$$H(z) = \frac{z}{\left(z - 0.8e^{j0.3\pi}\right)\left(z - 0.8e^{-j0.3\pi}\right)}$$

Poles @ $z = 0.8e^{\pm j0.3\pi}$



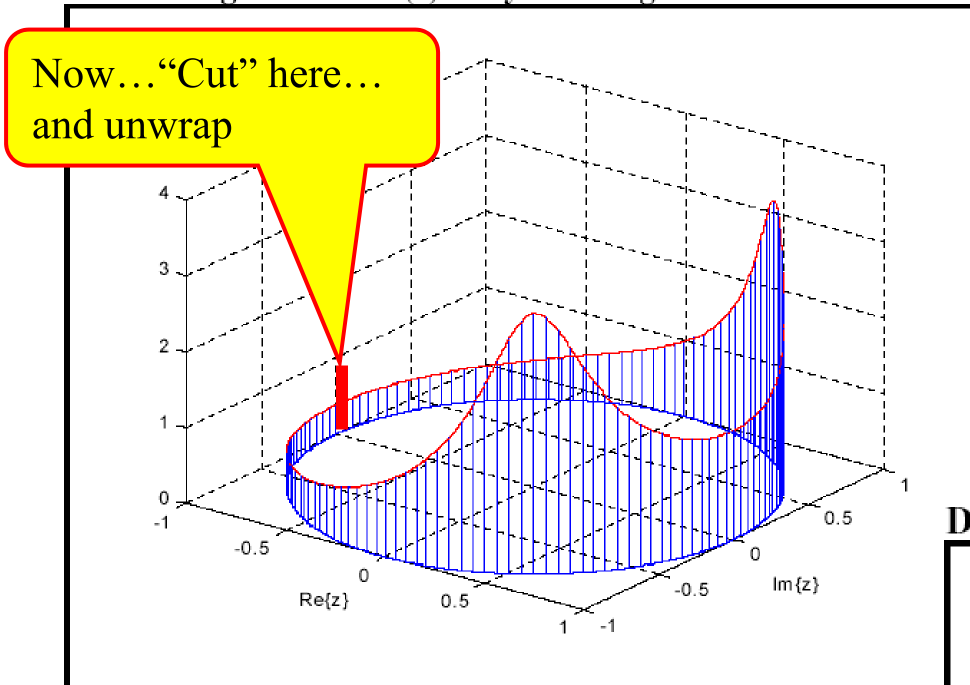


And we know that the Frequency Response is just the Transfer Function evaluated on the Unit Circle.

$$H(\Omega) = H(z)\big|_{z=e^{j\Omega}}$$

Now… plot just those values on the unit circle:

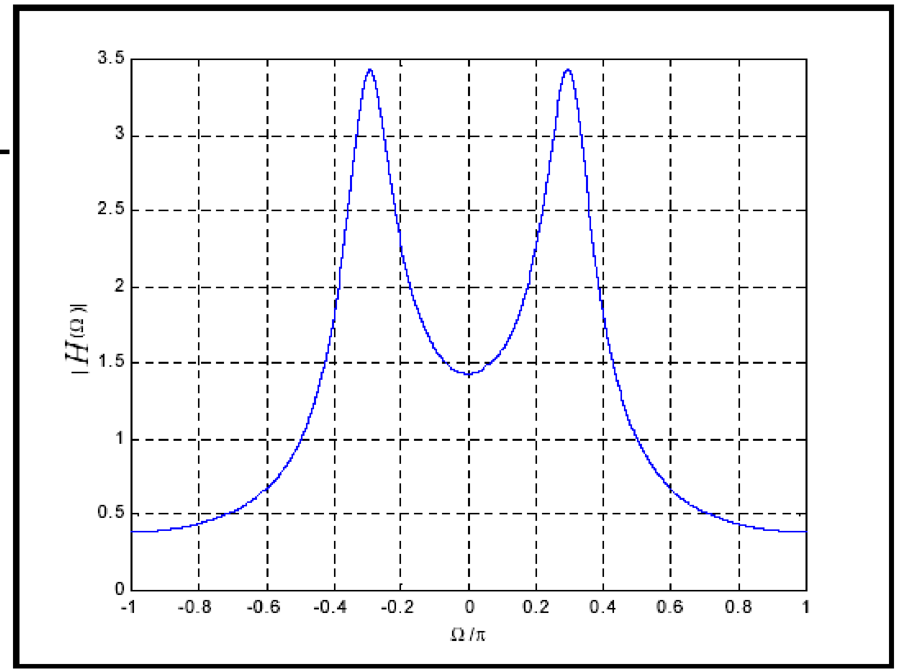
Plot of Magnitude of *H(z)* Only Showing Values on Unit Circle



This shows the Frequency Response *H*(Ω) where Ω is the angle around the unit circle… this explains why *H*(Ω) is a periodic function of Ω

This shows after it has been “cut and unwrapped”… and plotted on the Ω axis:

DTFT = ZT on Unit Circle



Effect of Poles & Zeros on Frequency Response of DT filters

Note: Including a pole or zero at the origin …

…doesn't change the magnitude but does change the phase

Placing a zero at $\pm\pi$…

…makes $|H(\pi)| = 0$

Placing more zeros/poles…

… gives sharper transitions.

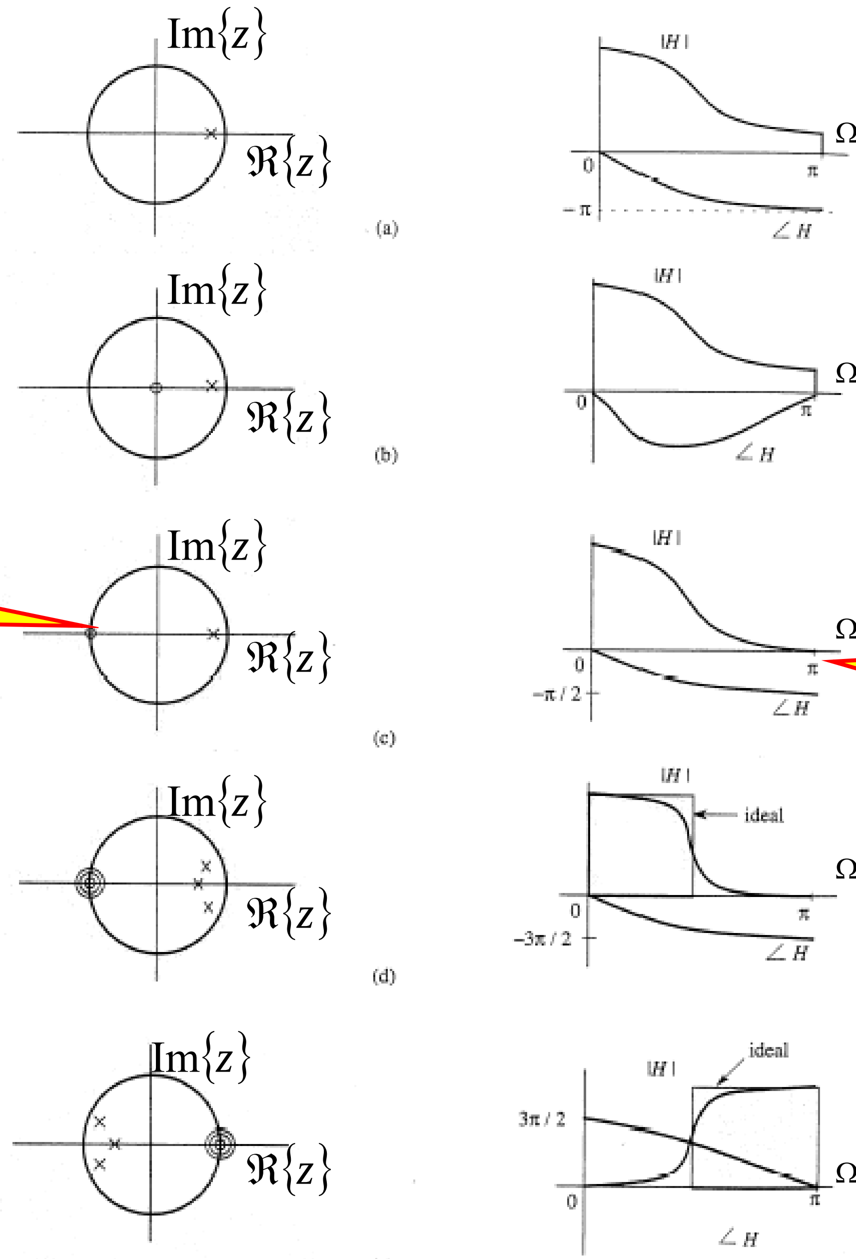
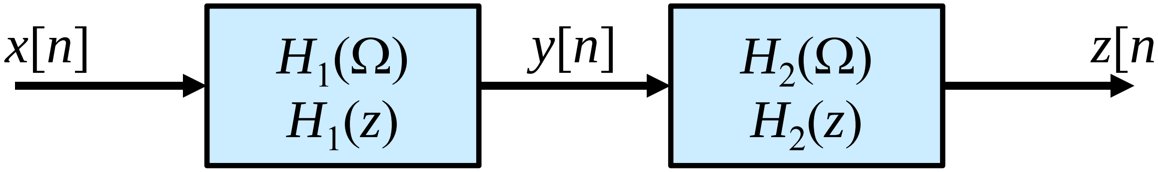


Figure from B.P. Lathi, Signal Processing and Linear Systems

Cascade of Systems

Suppose you have a "cascade" of two systems like this:



$$Y(\Omega) = H_1(\Omega) X(\Omega) \qquad Z(\Omega) = H_2(\Omega) Y(\Omega)$$
$$Y(z) = H_1(z) X(z) \qquad Z(z) = H_2(z) Y(z)$$

$$Z(\Omega) = H_2(\Omega) H_1(\Omega) X(\Omega)$$
$$Z(z) = H_2(z) H_1(z) X(z)$$

Thus, the ***overall*** frequency response/transfer function is the product of those of each stage:

$$H_{total}(\Omega) = H_1(\Omega) H_2(\Omega)$$
$$H_{total}(z) = H_1(z) H_2(z)$$

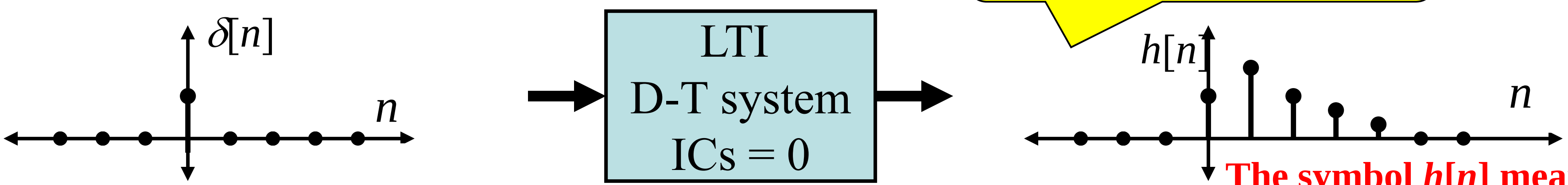
Obviously, this generalizes to a cascade of N systems:

$$H_{total}(\Omega) = H_1(\Omega) H_2(\Omega) \cdots H_N(\Omega)$$
$$H_{total}(z) = H_1(z) H_2(z) \cdots H_N(z)$$

Impulse Response of System

Sometimes looking at how a system responds to the impulse function (i.e., delta sequence) $\delta[n]$ can tell much about a system. Hitting a system with $\delta[n]$ is lot like ringing a bell to hear how it sounds…

Note: If system is causal, then $h[n] = 0$ for $n < 0$



The symbol $h[n]$ means "the impulse response".

Noting that the ZT of $\delta[n] = 1$ and using the properties of the transfer function and the Z transform:

$$h[n] = Z^{-1}\left\{H(z)Z\left\{\delta[n]\right\}\right\}$$

$$h[n] = Z^{-1}\left\{H(z)\right\}$$

$$h[n] = IDTFT\left\{H(\Omega)\right\}$$

From PFE and Poles/Zeros we see that a TF like this: $H(z) = z^{(N-M)}\dfrac{B(z)}{A(z)}$

…will have an impulse response with terms like this:

$$h[n] = k_1 p_1^n u[n] + k_2 p_2^n u[n] + \cdots + k_N p_N^n u[n]$$

Some simplifying assumptions made here!

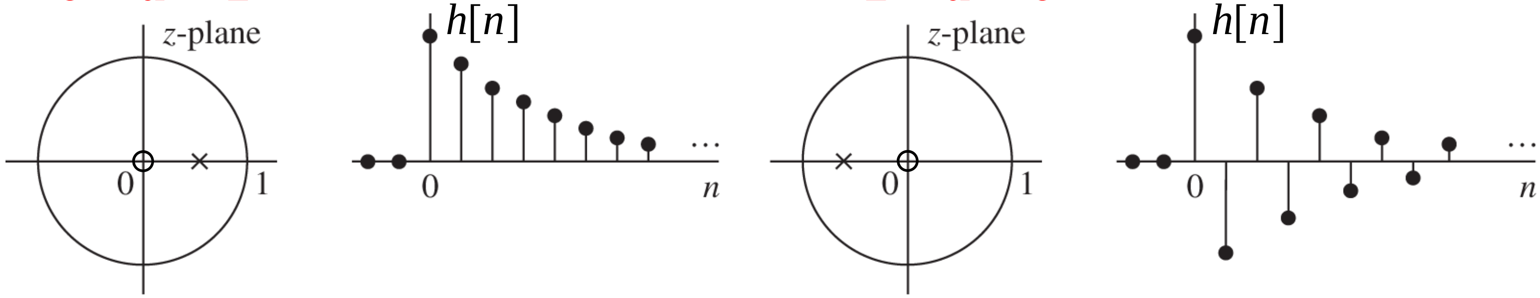
Now… we almost always want this to decay (like a bell!): all poles $|p_i| < 1$

From our knowledge of poles and the ZT we can now visualize the impulse response $h[n]$ of some simple systems:

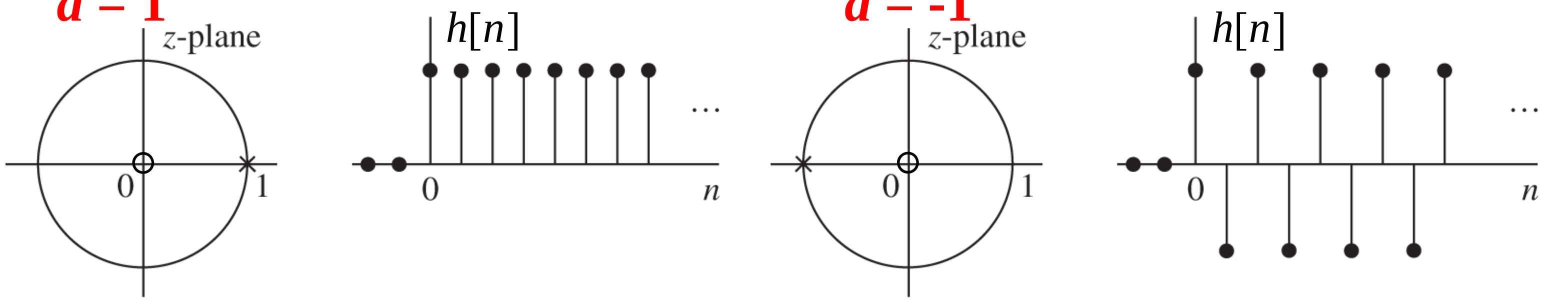
$$H(z) = \frac{z}{z-a}$$

Book doesn't show zeros!!

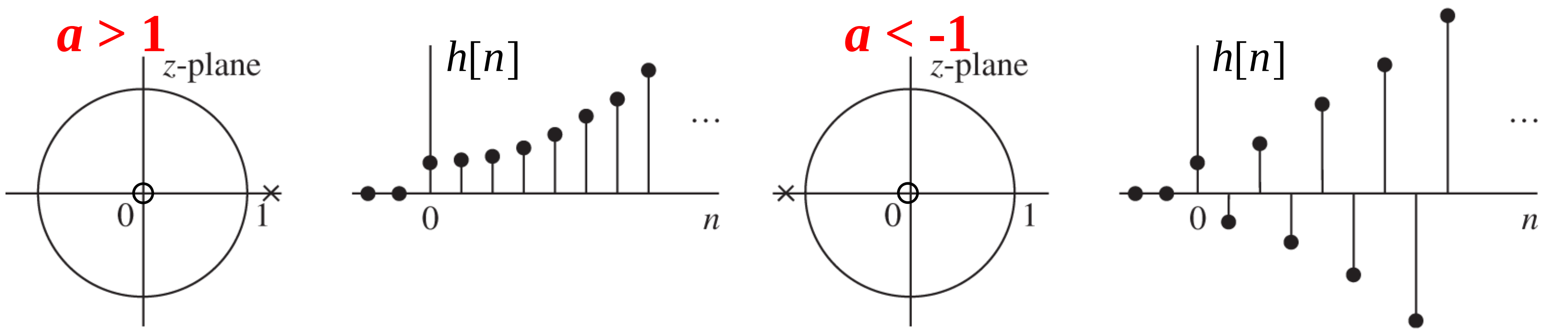
0 < *a* < 1 **-1 < *a* < 0**



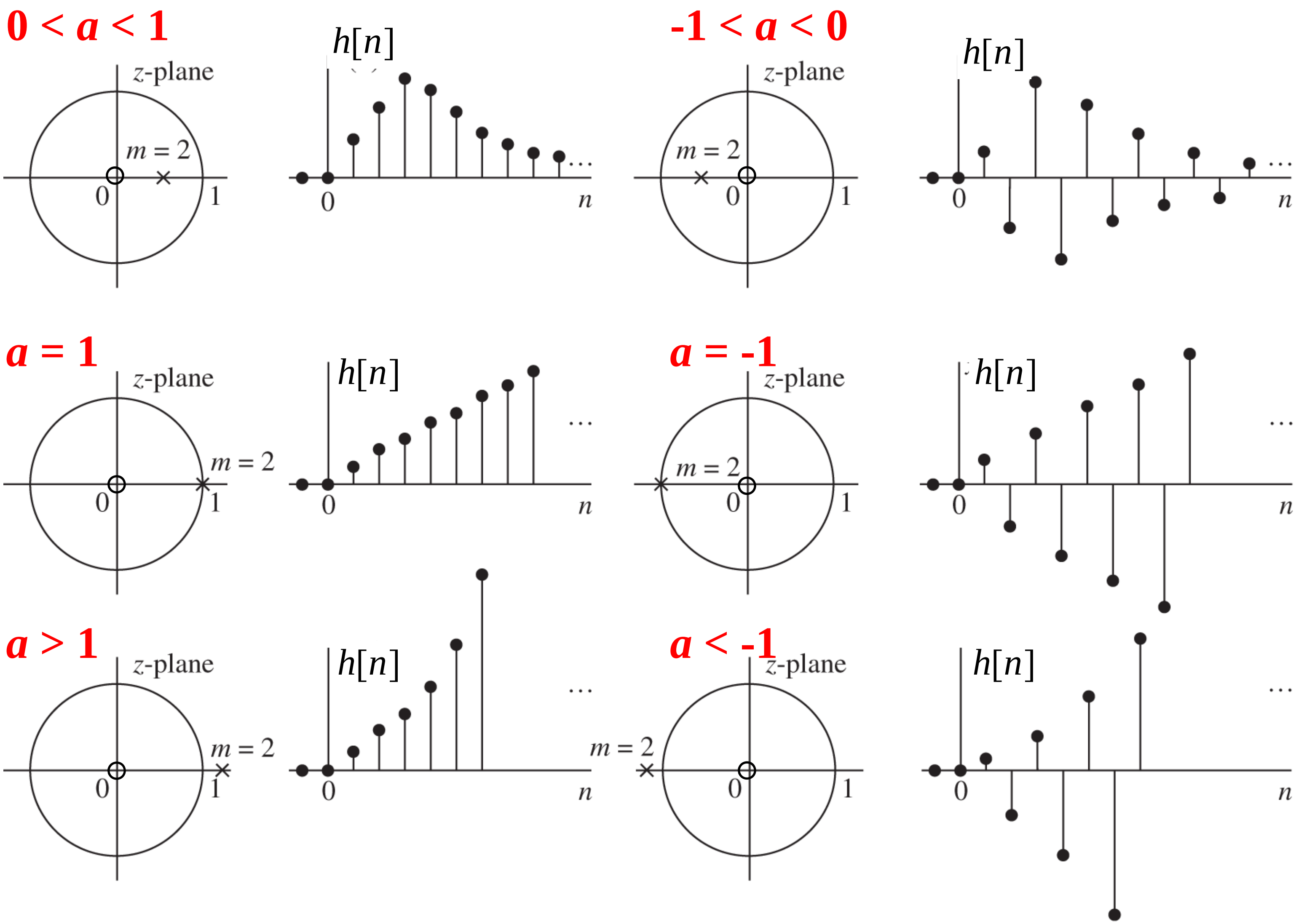
***a* = 1** ***a* = -1**



***a* > 1** ***a* < -1**

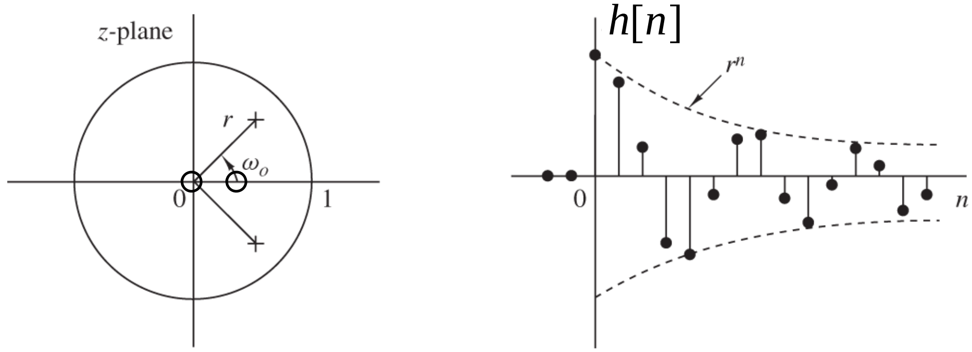


$$H(z) = \frac{z}{(z-a)^2}$$

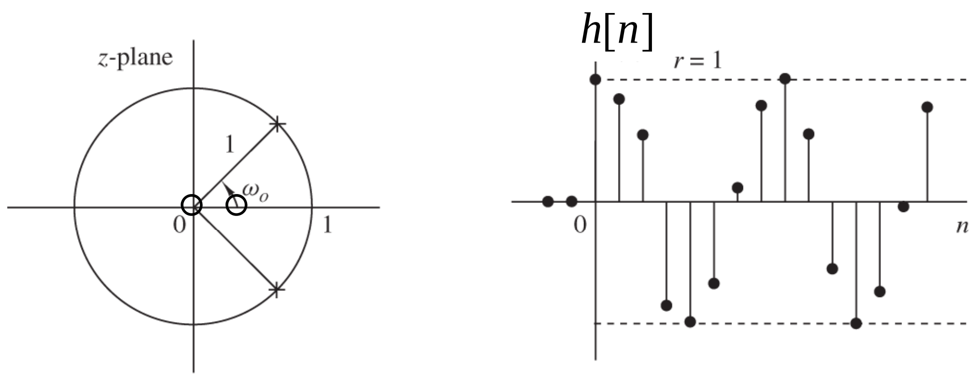

0 < a < 1
z-plane
m = 2
0
1
h[n]
0
n
-1 < a < 0
z-plane
m = 2
0
1
h[n]
0
n
a = 1
z-plane
m = 2
0
1
h[n]
0
n
a = -1
z-plane
m = 2
0
1
h[n]
0
n
a > 1
z-plane
m = 2
0
1
h[n]
0
n
a < -1
z-plane
m = 2
0
1
h[n]
0
n

$$H(z) = \frac{z[z - r\cos(\omega_o)]}{z^2 - 2r\cos(\omega_o)z + r^2}$$

$0 < r < 1$



$r = 1$



$r > 1$