BINGHAMTON UNIVERSITY
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EEO 401
Digital Signal Processing

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Note Set #7

- Properties of Z-Transform (Two-Sided Version)
- Reading Assignment: Sect. 3.2 of Proakis & Manolakis

These properties are very similar to those for CTFT, DTFT, and LT that you should have seen before…

Mostly they are helpful in deriving results and understanding more complicated ZT results.

The textbook provides many examples of how to apply them

Linearity: The ZT of a linear combination of two signals is the same linear combination of their individual ZT results, with a ROC that is the intersection of the two individual ROCs.

$$x_1[n] \leftrightarrow X_1^z(z) \text{ w/ } ROC_1$$

$$x_2[n] \leftrightarrow X_2^z(z) \text{ w/ } ROC_2$$

$$\Downarrow$$

$$a_1 x_1[n] + a_2 x_2[n] \leftrightarrow a_1 X_1^z(z) + a_2 X_2^z(z) \text{ w/ } ROC = ROC_1 \cap ROC_2$$

Except when the resulting signal is finite-duration… then use the ROC rules for finite-duration signals!

Time Shift: The ZT of a linear combination of two signals is the same linear combination of their individual ZT results, with a ROC that is the intersection of the two individual ROCs.

$$x[n] \leftrightarrow X^z(z) \text{ w/ } ROC_x$$

$$\boxed{y[n] = x[n-k] \quad \leftrightarrow \quad Y^z(z) = z^{-k} X^z(z)} \quad \text{w/ } ROC_y = \begin{cases} ROC_x \setminus \{z=0\}, & k>0 \\ ROC_x \setminus \{z=\infty\}, & k<0 \end{cases}$$

Example: $x[n] = \left\{\underset{\uparrow}{1}, 2, 5, 7, 0, 1\right\}$ $\qquad y[n] = x[n-2] = \left\{\underset{\uparrow}{0}, 0, 1, 2, 5, 7, 0, 1\right\}$

$$Z\{x[n]\} = \cdots + 0z^1 + \boxed{1z^0 + 2z^{-1} + 5z^{-2} + 7z^{-3} + 0z^{-4} + 1z^{-5}} + 0z^{-6} + \cdots$$

$$= 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$$

$$Z\{y[n]\} = \cdots + 0z^1 + 0z^0 + 0z^{-1} + \boxed{1z^{-2} + 2z^{-3} + 5z^{-4} + 7z^{-5} + 0z^{-6} + 1z^{-7}} + 0z^{-8} + \cdots$$

$$= 1z^{-2} + 2z^{-3} + 5z^{-4} + 7z^{-5} + z^{-7}$$

$$= z^{-2}\left[1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}\right]$$

Scaling in z Domain:

$$x[n] \leftrightarrow X^z(z) \text{ w/ } ROC_x : R_1 < |z| < R_2$$

$$y[n] = a^n x[n] \quad \leftrightarrow \quad Y^z(z) = X^z(a^{-1}z)$$

Valid for any constant *a*, real or complex

Just "scales" the ROC!

$$\text{w/ } ROC_y : |a|R_1 < |z| < |a|R_2$$

Time Reversal:

$$x[n] \leftrightarrow X^z(z) \text{ w/ } ROC_x : R_1 < |z| < R_2$$

$$y[n] = x[-n] \quad \leftrightarrow \quad Y^z(z) = X^z(z^{-1})$$

Essentially "flips" the ROC!

$$\text{w/ } ROC_y : \frac{1}{R_2} < |z| < \frac{1}{R_1}$$

Multiply by *n*:

$$x[n] \leftrightarrow X^z(z) \text{ w/ } ROC_x : R_1 < |z| < R_2$$

$$y[n] = nx[n] \quad \leftrightarrow \quad Y^z(z) = -z\frac{dX^z(z)}{dz}$$

$$\text{w/ } ROC_y = ROC_x$$

Convolution:

$$x_1[n] \leftrightarrow X_1^z(z) \text{ w/ } ROC_1$$

$$x_2[n] \leftrightarrow X_2^z(z) \text{ w/ } ROC_2$$

Could be bigger than intersection when a one ZT is zero where the other does not converge

$$y[n] = x_1[n] * x_2[n] \quad \leftrightarrow \quad Y^z(z) = X_1^z(z) X_2^z(z)$$

$$ROC \supset ROC_1 \cap ROC_2$$

ZT, CTFT, LT, & DTFT: …**Convolution Transforms to Multiplication!!!**

LTI System Property

$$x[n] \rightarrow \boxed{h[n],\ H(z)} \rightarrow y[n] = Z^{-1}\{Y^z(z)\}$$

$X^z(z)$

$$Y^z(z) = X^z(z) H^z(z)$$

Transfer Function:
$H^z(z) = Z\{h[n]\}$

Note how similar this is to what we saw for DTFT:

$$x[n] \rightarrow \boxed{h[n],\ H(\Omega)} \rightarrow y[n] = \mathscr{F}^{-1}\{Y^f(\Omega)\}$$

$X^f(\Omega)$

$$Y^f(\Omega) = X^f(\Omega) H^f(\Omega)$$

Frequency Response:
$H^f(\Omega) = \text{DTFT}\{h[n]\}$

Inversion of the z-Transform

Once we have the ZT of the output $Y^z(z) = H^z(z)\, X^z(z)$ we need to convert it back into $y[n]$ to find out what we expect the output to look like…

Section 3.4 discusses various ways to do this… the most important being the use of partial fraction expansion. See my coverage in my Signals & Systems course for details on how to do that.

Discrete-Time System Relationships