BINGHAMTON UNIVERSITY
State University of New York

EEO 401
Digital Signal Processing

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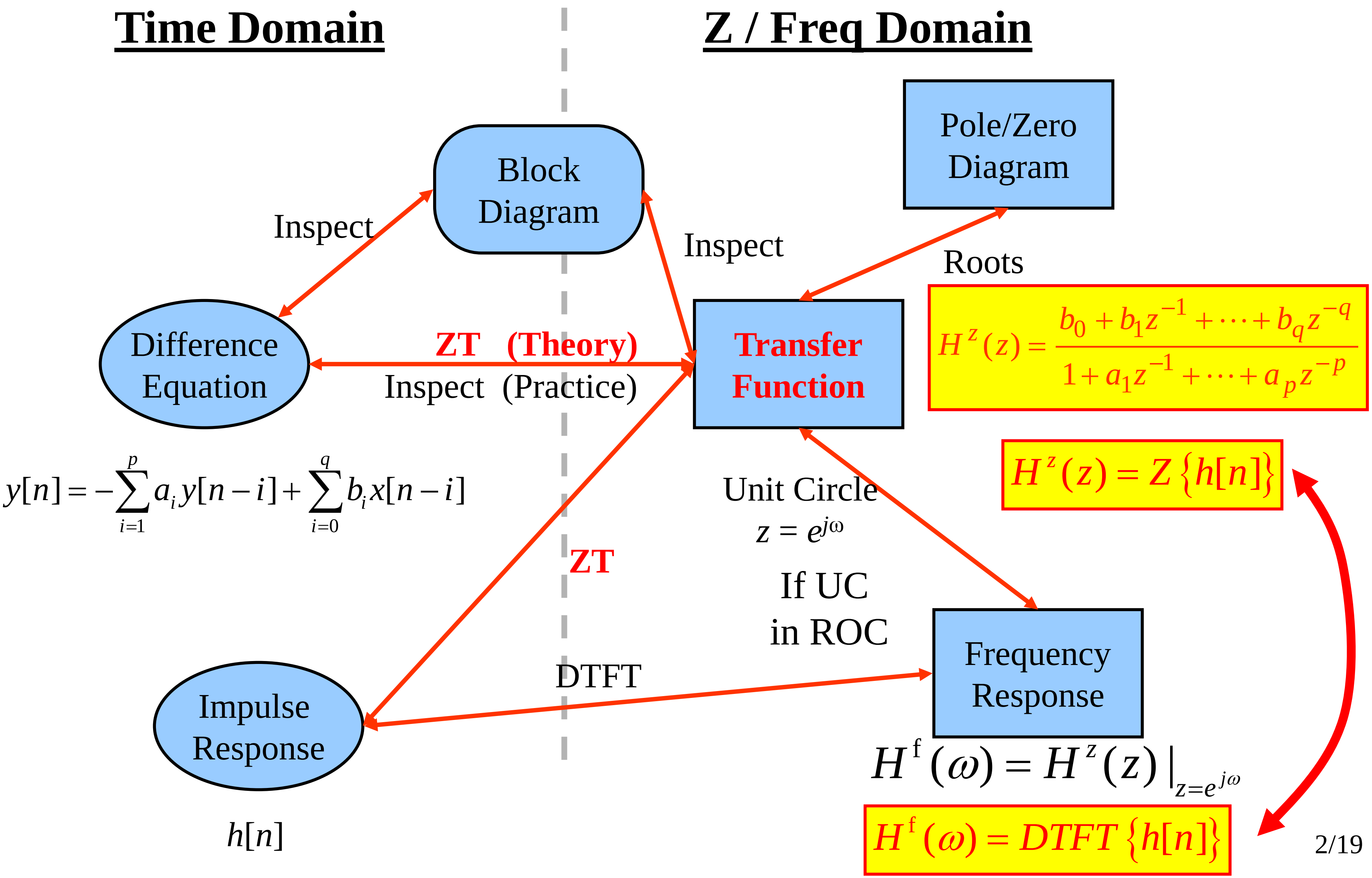
Note Set #6

- Intro to Z-Transform
- Reading Assignment: Sect. 3.1 of Proakis & Manolakis

Discrete-Time System Relationships



Transforms for Systems

Proakis & Manolakis don't use this superscript Notation. I borrowed it from Porat's DSP Book

DTFT of Impulse Reponse = Frequency Response of System

$$H^{f}(\omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$$

Discrete **T**ime **F**ourier **T**ransform

ZT of Impulse Reponse = Transfer Function of System

$$H^{z}(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

For Zero ICs Only

$$Y^{f}(\omega) = H^{f}(\omega)X^{f}(\omega)$$

Same Idea!!

$$Y^{z}(z) = H^{z}(z)X^{z}(z)$$

Q: If both FR & TF reduce system analysis to multiplication why do we need both??

A: TF is a generalization of FR… it allows

- Greater insight into system characteristics… it is a more powerful tool!!
- Application to wider class of systems (namely… unstable systems)
- Easily extended to allow handling ICs (more of a "Controls" topic than DSP)

Q: If TF is more powerful why even consider FR?

A: FR provides a better connection back to "reality"

- Based on Fourier ideas… connects back to how systems modify sinusoids

DTFT is also used to analyze signals… ZT is not really a useful signal analysis tool

Z-Transform Definition

Given a D-T signal $x[n]$ $-\infty < n < \infty$ the DTFT of $x[n]$ is given by:

$$DTFT: X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$ (Periodic in ω with period 2π)

Issue: The sum in the DTFT doesn't converge for some signals… In particular, for the impulse response of an unstable system.

So, to extend applicability to such cases we include a decay into the transform:

$$e^{-j\omega n} \quad vs. \quad \alpha^{-n}e^{-j\omega n} = (\alpha e^{j\omega})^{-n} \triangleq z^{-n}$$

Note… these are complex numbers with unit magnitude

Controls decay of summand

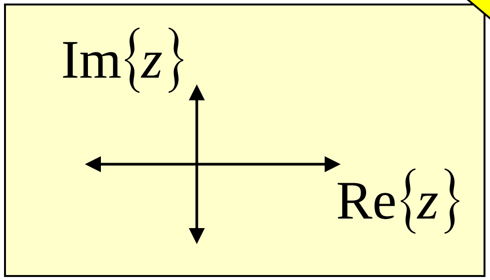
Note… z is a complex number anywhere on plane

Z-Transform (Two-Sided)

$$X^z(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \qquad z \text{ is complex-valued}$$

The "one-sided" ZT is used more in control systems… less in DSP.

So… the Z-Transform gives a complex-valued function on a complex plane called the "z-plane"

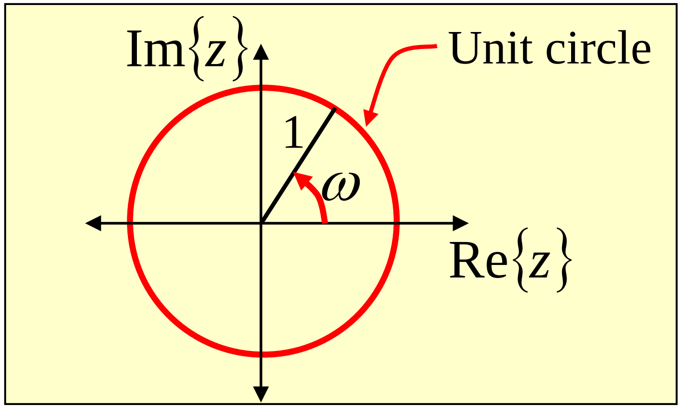


Plot Magnitude on complex z-plane. (Rarely actually plot phase on z-plane)

For the Z-Transform we'll need to divide the plane into two parts:

- those values of z inside the unit circle
- those values of z outside the unit circle

"**Unit Circle**" = all z such that $|z| = 1$, i.e. all $z = e^{j\omega}$



Region of Convergence (ROC)

Remember that the whole motivation for the Z-Transform was to give a transform that "converges" for a wider class of signals than does the DTFT

$$DTFT: X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$ZT: \quad X^z(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \qquad\qquad z^{-n} \triangleq (\alpha e^{j\omega})^{-n}$$

So… given a signal for which the DTFT does not converge the ZT will converge for only certain values of *z* (clearly in this case it can't converge for those *z* with $|z| = 1$!!!)

So… in general then… for each signal $x[n]$ there will only be certain values of *z* for which the ZT summation converges (or sometimes none at all!).

Define: ROC = Set of all *z* values for which the ZT sum converges ($|X(z)| < \infty$)

Each signal has its own region of convergence.

Example of Finding the ZT: Unit Impulse Sequence

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

$$Z\{\delta[n]\} = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n}$$

$$= \cdots + 0 \times z^{2} + 0 \times z^{1} + 1 \times z^{0} + 0 \times z^{-1} + 0 \times z^{-2} + \cdots$$

$$= 1$$

ROC = all complex #'s

There's really only one term here so convergence is not an issue!

$$\delta[n] \quad \leftrightarrow \quad 1 \quad \text{ROC} = \{\text{all } z\}$$

This result and many others are on Table of Z Transforms available on my website…

Z Transform of Finite-Duration Signals

For finite-duration signals the ZT is easy to find and the ROC is also easy to find because there are not infinitely many terms in the summation!

Example: $x[n] = \{\underset{\uparrow}{1}, 2, 5, 7, 0, 1\}$

Finite Duration & Causal

$$Z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \cdots + 0z^{1} + 1z^{0} + 2z^{-1} + 5z^{-2} + 7z^{-3} + 0z^{-4} + 1z^{-5} + 0z^{-6} + \cdots$$

$$= 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$$

At z = 0 we have terms that have 1/0

ROC = entire z plane except $z = 0$

Example: $x[n] = \{1, 2, \underset{\uparrow}{5}, 7, 0, 1\}$

Finite Duration & "Bi-Directional"

$$Z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= 1z^{2} + 2z^{1} + 5 + 7z^{-1} + z^{-3}$$

ROC = entire z plane except $z = 0$ and $z = \infty$

Example: $x[n] = \left\{1, 2, 5, 7, 0, \underset{\uparrow}{1}\right\}$

Finite Duration &
Anti-Causal

$$Z\left\{x[n]\right\} = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= 1z^5 + 2z^4 + 5z^3 + 7z^2 + 1$$

ROC = entire z plane

except $z = \infty$

Summary of ROC for Finite-Duration Signals

Type of Finite-Duration Signal	**ROC**
Causal	All z except $z = 0$
Anti-Causal	All z except $z = \infty$
Bi-Directional	All z except $z = \infty$ & $z = 0$

Z Transform of Infinite-Duration Signals

For infinite-duration signals finding the ZT is harder and finding the ROC requires some attention!

Example of Finding the ZT: Unit Step $u[n]$

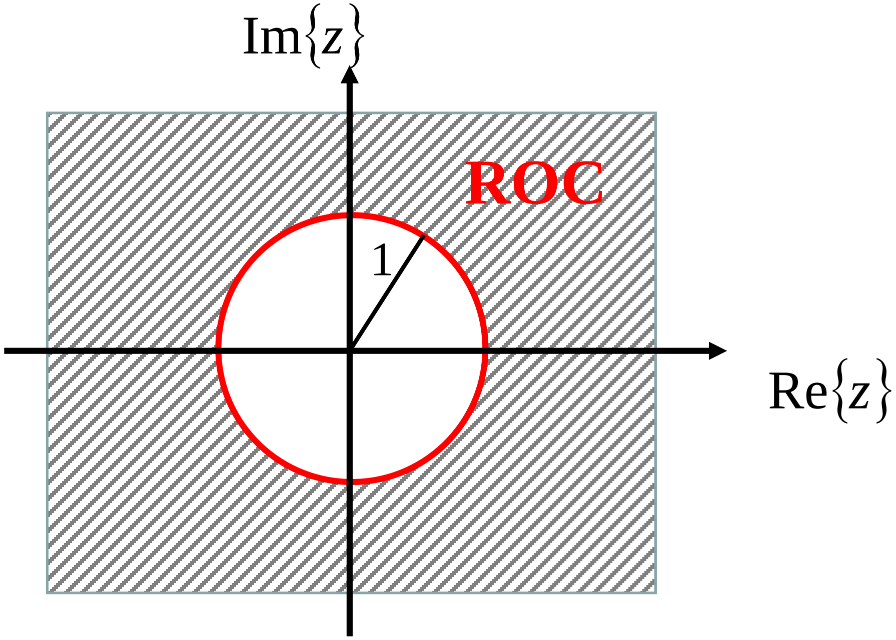
$$U^z(z) = \sum_{n=0}^{\infty} u[n] z^{-n} = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

ROC = all z such that $|z| > 1$

Using standard result for "geometric sum"… converges when $|z|>1$

$$u[n] \leftrightarrow \frac{z}{z-1} = \frac{1}{1-z^{-1}}$$

$$\text{ROC} = \text{all } z \text{ w/ } |z| > 1$$



Example of Finding the ZT: Causal Exponential

$$x[n] = a^n u[n]$$

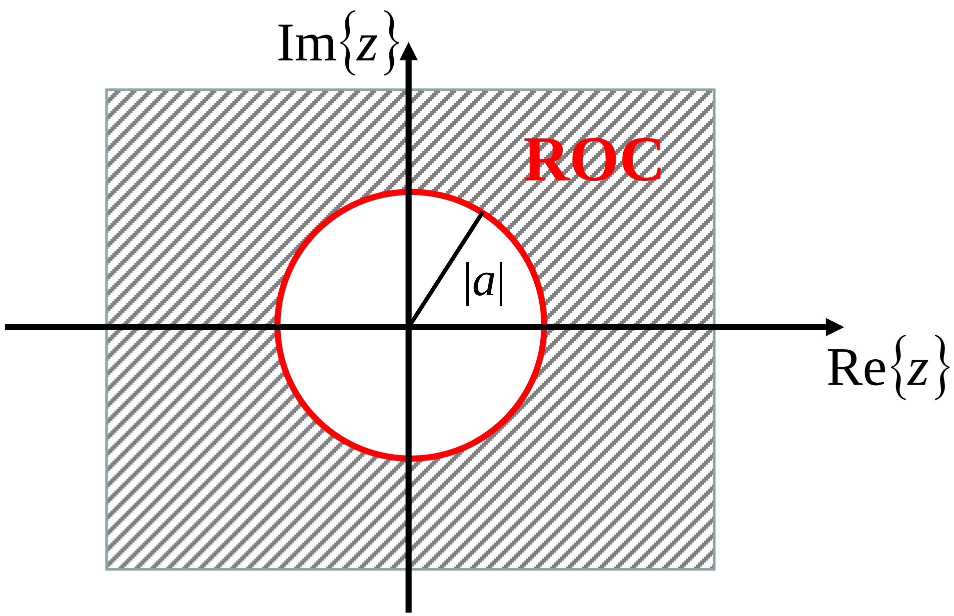
Using standard result for "geometric sum"… converges if $|a/z| < 1$

Again using geometric sum: $$X^z(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(\tfrac{a}{z}\right)^n = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

ROC = all z such that $|z| > |a|$

Note: The example for $u[n]$ is just a special case of this example

$$a^n u[n] \leftrightarrow \frac{z}{z-a} = \frac{1}{1-az^{-1}} \quad \text{ROC = all } z \text{ such that } |z| > |a|$$



General Result for ROC of Causal Signal

$$X^z(z) = \sum_{n=0}^{\infty} x[n]z^{-n} = x[0]z^0 + x[1]z^{-1} + x[2]z^{-2} + x[3]z^{-3} + \cdots$$

Now use $z \triangleq \alpha e^{j\omega}$

$$X^z(z) = x[0] + x[1]\alpha^{-1}e^{-j\omega} + x[2]\alpha^{-2}e^{-j2\omega} + x[3]\alpha^{-3}e^{-j3\omega} + \cdots$$

With a little thought we can guess that it should be the value of α that will be important to ensure that the terms in this sum get "small enough" "fast enough" so that the summation converges.

And… we can actually show that mathematically:

$$\left|X^z(z)\right| = \left|\sum_{n=0}^{\infty} x[n]z^{-n}\right| \overset{|a+b| \le |a|+|b|}{\le} \sum_{n=0}^{\infty} \left|x[n]z^{-n}\right| = \sum_{n=0}^{\infty} \left|x[n]\right|\left|z^{-n}\right|$$

$$= \sum_{n=0}^{\infty} \left|x[n]\right|\left|\alpha^{-n}\right|\underbrace{\left|e^{-j\omega n}\right|}_{=1} = \sum_{n=0}^{\infty} \frac{\left|x[n]\right|}{\alpha^n}$$

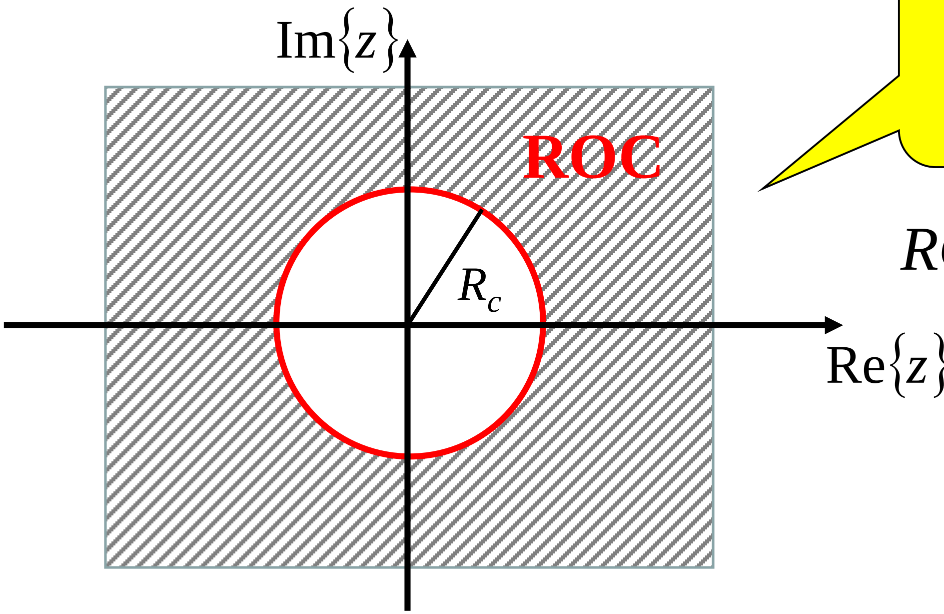
$z \triangleq \alpha e^{j\omega}, \alpha \ge 0$

So only α (magnitude of z) matters when finding z values to ensure $|X(z)| < \infty$

So… $\left|X^z(z)\right| \leq \sum_{n=0}^{\infty} \frac{|x[n]|}{\alpha^n}$

And… the slower $|x[n]|$ decays the faster we need $1/\alpha^n$ to decay. Note that if we find some value of α that works, all values larger than it will also work.

Thus, if there *is* a ROC then there is some smallest α that works and all *z* with magnitudes larger than it will be in the ROC.



So ROC for causal signals is the region <u>outside</u> some circle.

$$ROC = \{ z \mid |z| > R_c \}$$

Note that the result for a finite-duration <u>causal</u> signal is consistent with this, where $R_c = 0$

General Result for ROC of Anti-Causal Signal

$$X^z(z) = \sum_{n=-\infty}^{0} x[n]z^{-n} = x[0]z^0 + x[-1]z^1 + x[-2]z^2 + x[-3]z^3 + \cdots$$

Now use $z \triangleq \alpha e^{j\omega}$

$$X^z(z) = x[0] + x[-1]\alpha^1 e^{j\omega} + x[-2]\alpha^2 e^{j2\omega} + x[-3]\alpha^3 e^{j3\omega} + \cdots$$

And… we can actually show that mathematically:

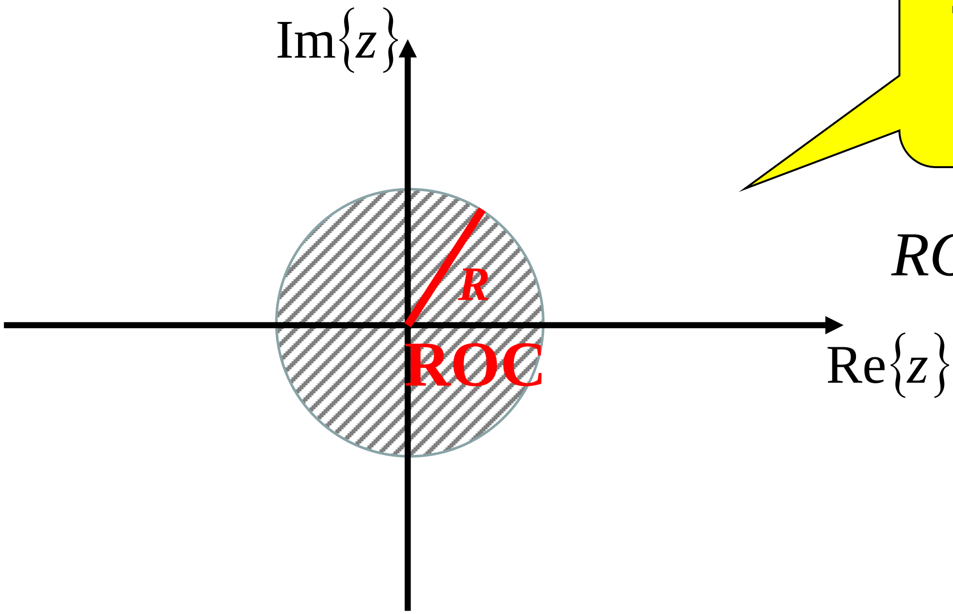
$$\left|X^z(z)\right| = \left|\sum_{n=-\infty}^{0} x[n]z^{-n}\right| = \left|\sum_{n=0}^{\infty} x[-n]z^{n}\right| \le \sum_{n=0}^{\infty}\left|x[-n]z^{n}\right| = \sum_{n=0}^{\infty}\left|x[-n]\right|\left|z^{n}\right|$$

$$= \sum_{n=0}^{\infty}\left|x[-n]\right|\left|\alpha^{n}\right|\underbrace{\left|e^{j\omega n}\right|}_{=1} = \sum_{n=0}^{\infty}\left|x[-n]\right|\alpha^{n}$$

So… $\left|X^z(z)\right| \le \sum_{n=0}^{\infty} \left|x[-n]\right| \alpha^n$

And… the slower $|x[-n]|$ decays the faster we need α^n to decay. Note that if we find some value of α that works, all values smaller than it will also work.

Thus, if there *is* a ROC then there is some largest α that works and all *z* with smalller larger than it will be in the ROC.



So ROC for anti-causal signals is the region inside some circle.

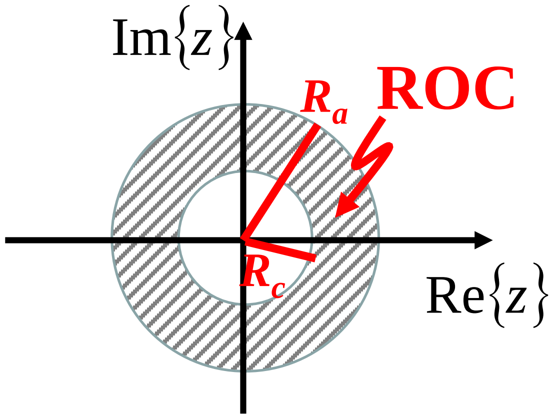
$$ROC = \left\{ z \,\middle|\, |z| < R_a \right\}$$

Note that the result for a finite-duration anti-causal signal is consistent with this, where $R_a = \infty$

General Result for ROC

Any general signal can be decomposed into one part that is causal and one part that is anti-causal. So it is no surprise that in general the ROC would be a ring-shaped (annular) region:

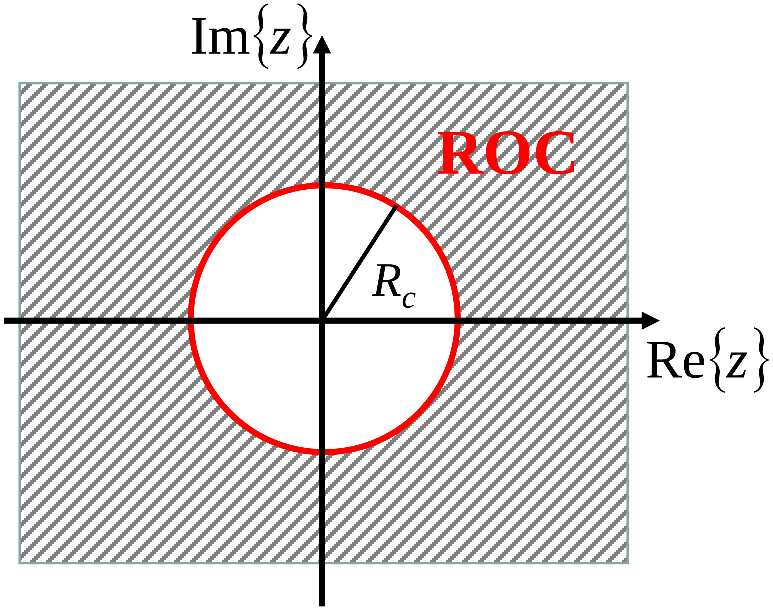
$$ROC = \left\{ z \mid R_c < |z| < R_a \right\}$$



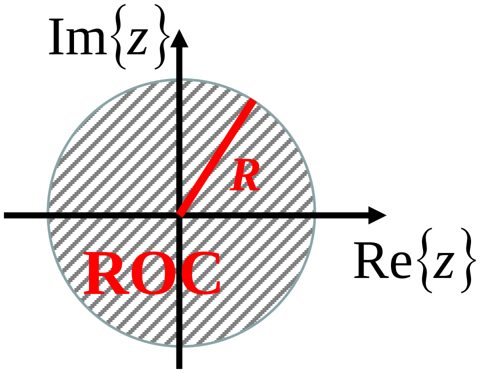
Note that if for the given signal we have that Ra < Rc then the two individual regions (causal & anti-causal) do not overlap and there is no region of *z* such that both parts converge. Then we say that there is no ROC for that signal and we can not apply the ZT to that signal.

Summary of Results for ROC

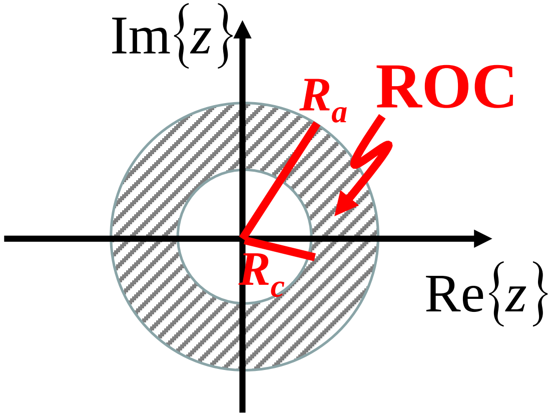
Causal Signal



Anti-Causal Signal



Bi-Directional Signal



Connection between ZT and DTFT:

If the ROC contains the UC then the ZT converges for $z = e^{j\omega}$ and that means that the DTFT exists. So then if we have the ZT we can easily convert it into a DTFT by replacing z by $e^{j\omega}$

Example:

$$x[n] = a^n u[n] \quad \leftrightarrow \quad X^z(z) = \frac{z}{z-a} = \frac{1}{1-az^{-1}} \quad \text{ROC = all } z \text{ such that } |z| > |a|$$

If $|a| < 1$ then ROC includes the UC and the DTFT of this signal is

$$X^z(z) = \frac{1}{1-az^{-1}} \quad \Rightarrow \quad X^f(\omega) = X^z(z)\Big|_{z=e^{j\omega}} = \frac{1}{1-az^{-1}}\Bigg|_{z=e^{j\omega}} = \frac{1}{1-ae^{-j\omega}}$$

$\delta[n-q], \quad q = \pm 1, \pm 2, \pm 3, \ldots$	$e^{-jq\Omega}, \quad q = \pm 1, \pm 2, \pm$
$a^n u[n], \quad \lvert a \rvert < 1$	$\frac{1}{1-ae^{-j\Omega}}, \quad \lvert a \rvert < 1$
$e^{j\Omega_o n}, \quad \Omega_o$ real	[illegible]

Matches
DTFT Result
for this signal

Discrete-Time System Relationships