EEO 401
Digital Signal Processing

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Note Set #34

- IIR Design – Characteristics of Common Analog Filters
- Reading: Sect. 10.3.4 & 10.3.5 of Proakis & Manolakis

Motivation

We've seenthat the Bilinear Transform provides an easy and effective way to take some analog filter design and transform it into a DT IIR filter design.

This viewpoint arose because when DT IIR filter designs were first being developed the understanding of analog filters was already well established. So… here we take a look at some of the existing analog filters that are commonly used with the Bilinear Transform to make DT IIR filters.

Here we focus on lowpass filters… in the next note set we'll learn how to transform these into other filter types.

- **Butterworth Filters**
  - Magnitude is monotonic with no ripples
  - "Maximally Flat"
- **Chebyshev Filters**
  - Type I: equiripple in passband & monotonic in stopband
  - Type II: monotonic in passband & equiripple in stopband
- **Elliptic Filters**
  - Ripple in passband & in stopband
- **Bessel Filters**
  - Linear phase in passband… but not preserved after bilinear transform!

Butterworth Filters

An N^{th} order Lowpass Butterworth filter is an all-pole filters with a frequency response satisfying the following on its magnitude ***squared*** response

$$\left|H^F(\Omega)\right|^2 = \frac{1}{1+\left(\Omega/\Omega_c\right)^{2N}}$$

Ω_c = -3dB freq
("cutoff freq.")

Note: Evaluating this at $\Omega = \Omega_c$ gives $\left|H^F(\Omega_c)\right|^2 = \frac{1}{2} = -3dB$

Converting this relationship into the *s*-form gives:

$$H^L(s)H^L(-s) = \frac{1}{1+\left(-s^2/\Omega_c^2\right)^N} \Rightarrow 1+\left(-s^2/\Omega_c^2\right)^N = 0$$

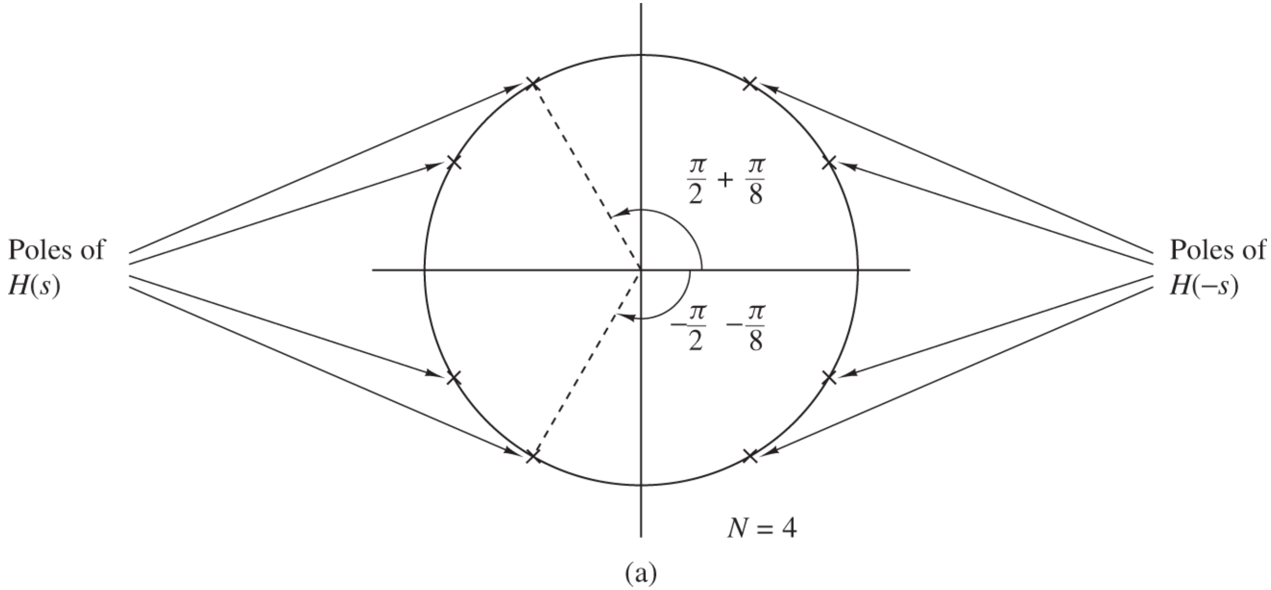
Describes poles of $H^L(s)$ & $H^L(-s)$

Poles are equally spaced according to

Poles equally spaced in angle at radius Ω_c

$$s_k = \Omega_c e^{j\pi/2} e^{j(2k+1)\pi/2N}, \quad k = 0,1,2,\ldots,N-1$$

See plots on next page



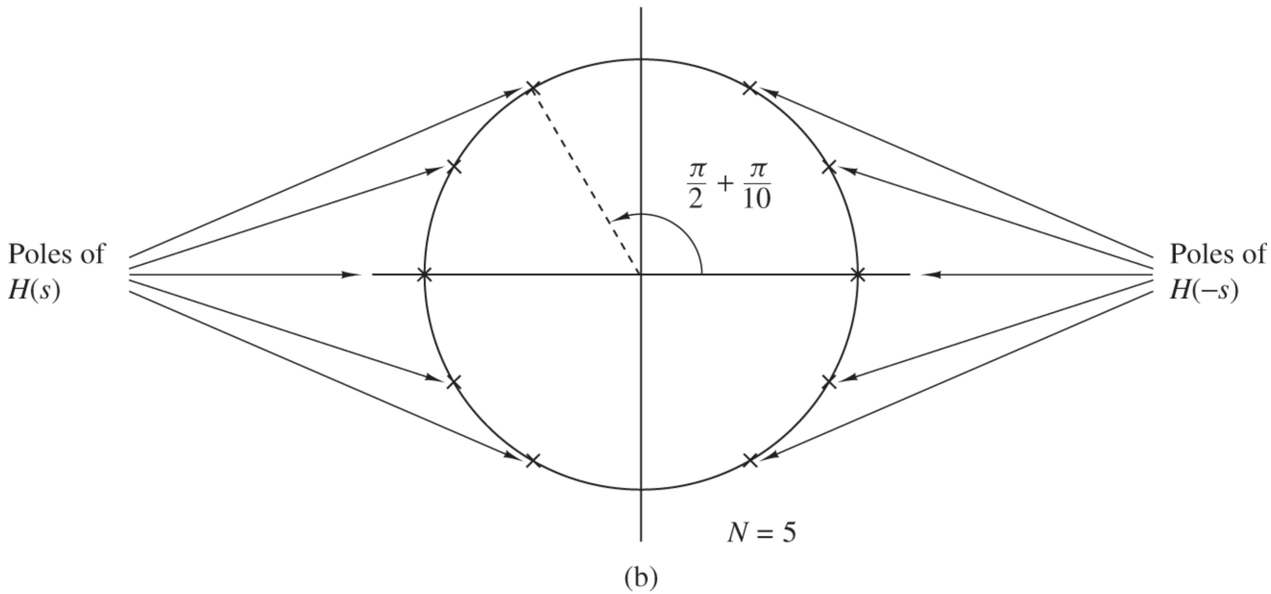


Figure 10.3.9 Pole positions for Butterworth filters.

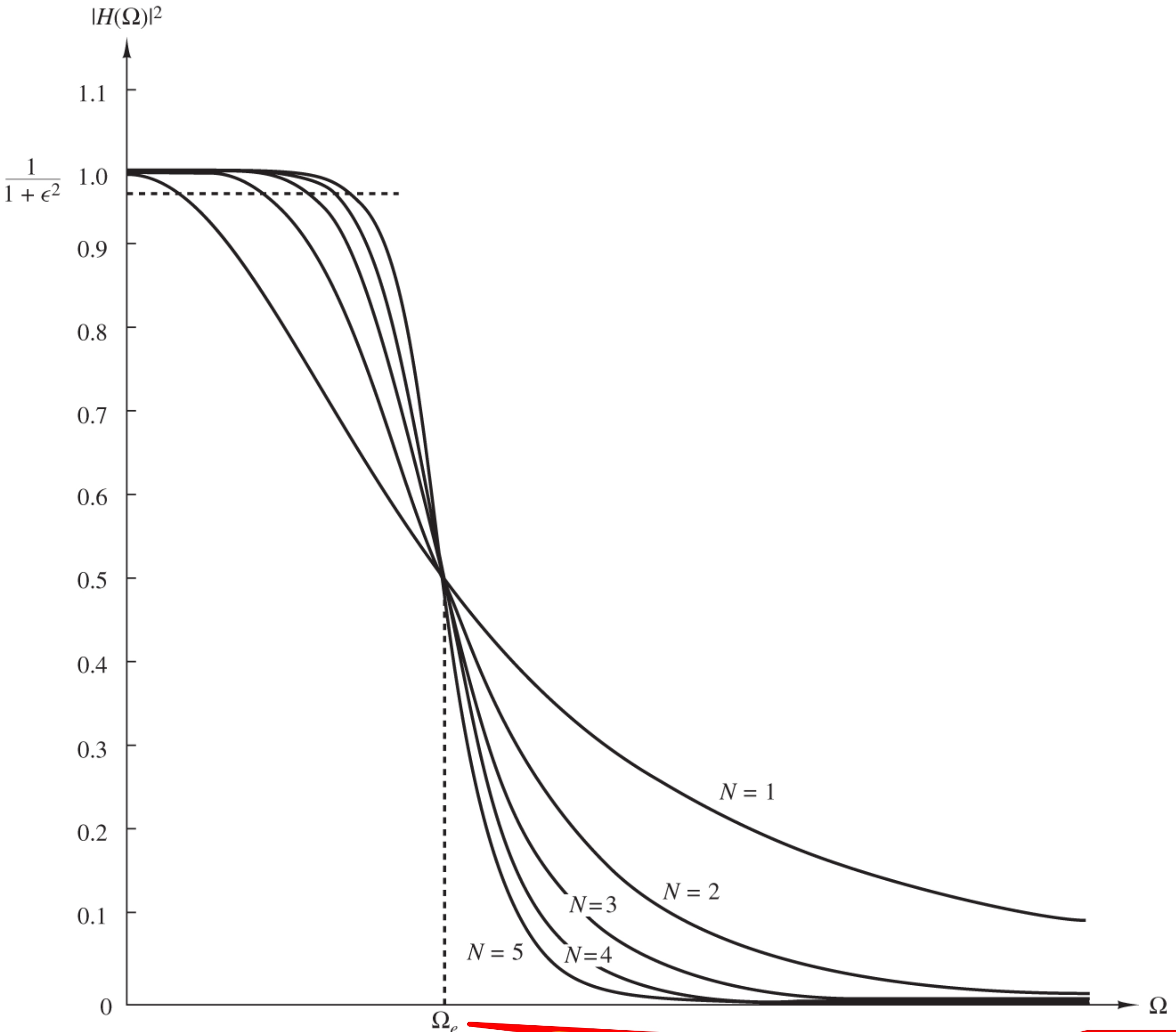


Figure 10.3.10 Frequency response of Butterworth filters.

Ω_c = -3dB freq
("cutoff freq.")

Specifying Butterworth Filters

We need only specify order N and cutoff frequency Ω_c (-3dB frequency)

Sometimes it is desirable to use alternative specs…

$$\left|H^F(\Omega)\right|^2 = \frac{1}{1+\left(\Omega/\Omega_c\right)^{2N}} = \frac{1}{1+\varepsilon^2\left(\Omega/\Omega_p\right)^{2N}}$$

Ω_p = passband edge

$$\Rightarrow \quad \left|H^F(\Omega_p)\right|^2 = \frac{1}{1+\varepsilon^2}$$

Ω_c is for $\varepsilon = 1$

For a given Ω_p and ε we can find the attenuation level at some specified stopband frequency Ω_s

$$\left|H^F(\Omega_s)\right|^2 == \frac{1}{1+\varepsilon^2\left(\Omega_s/\Omega_p\right)^{2N}} = \delta_2^2$$

Spec'd value at Ω_s

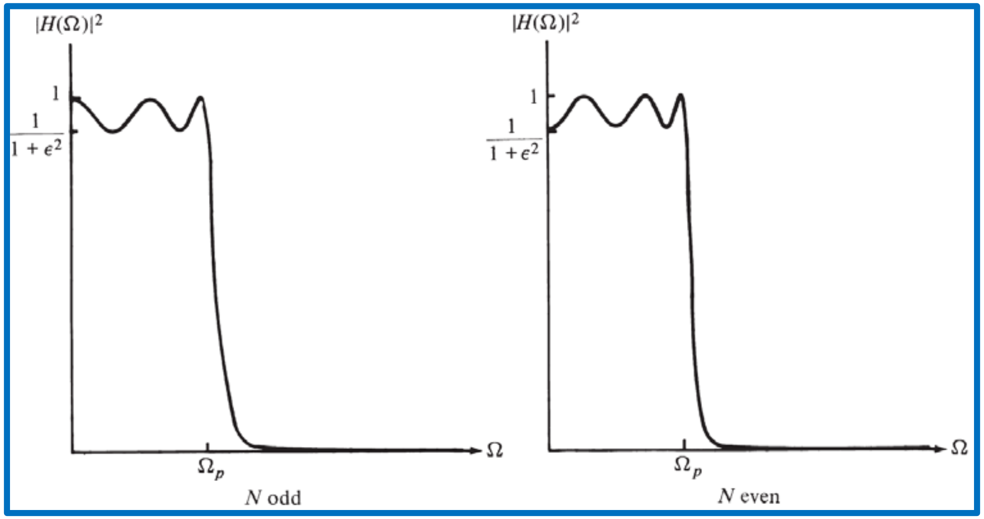
Order needed:

$$\Rightarrow \quad N = \frac{\log_{10}\left[\left(1/\delta_2^2\right)-1\right]}{2\log_{10}\left(\Omega_s/\Omega_c\right)} = \frac{\log_{10}\left(\delta/\varepsilon\right)}{2\log_{10}\left(\Omega_s/\Omega_p\right)}$$

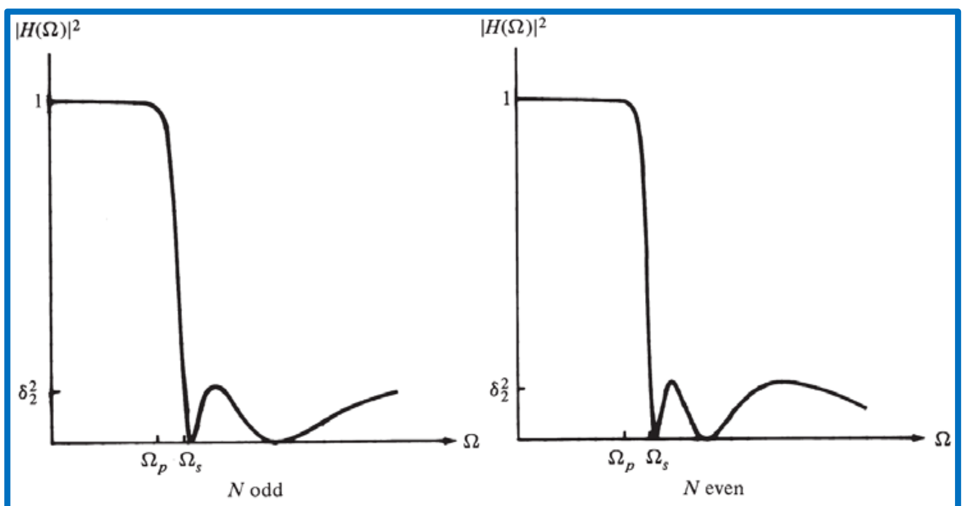
Example 10.3.6 shows how to use these to spec a Butterworth filter

Chebyshev Filters

- There are two types:
  - Type I: equiripple in passband & monotonic in stopband
  - Type II: monotonic in passband & equiripple in stopband



Type I



Type II

Type I Chebyshev lowpass filters are all-pole and are described by

$$\left|H^F(\Omega)\right|^2 = \frac{1}{1+\varepsilon^2 T_N^2\left(\Omega/\Omega_p\right)}$$

Sets Ripple Level

Type II Chebyshev lowpass filters are have poles and zeros and are described by

$$\left|H^F(\Omega)\right|^2 = \frac{1}{1+\varepsilon^2\left[\dfrac{T_N^2\left(\Omega_s/\Omega_p\right)}{T_N^2\left(\Omega_s/\Omega\right)}\right]}$$

Sets Ripple Level

Defined in terms of the "Chebyshev Polynomials"

$$T_N(x) = \begin{cases} \cos(N\cos^{-1}x), & |x| \le 1 \\ \cosh(N\cosh^{-1}x), & |x| > 1 \end{cases}$$

which can be found iteratively:

$$T_0(x) = 1$$

$$T_1(x) = x$$

$$T_k(x) = 2xT_{k-1}(x) - T_{k-2}(x), \quad k = 2, 3, 4, \ldots$$

As in Butterworth design… there are equations that relate the design to specs on Cuttoff & stopband frequencies, ripple, and stopband attenuation

Elliptic (or Cauer) Filters

This class of filters has both poles and zeros and has equiripple in both bands:

$$\left|H^F(\Omega)\right|^2 = \frac{1}{1+\varepsilon^2 U_N^2\left(\Omega/\Omega_p\right)}$$

where $U_N(x)$ is the Jacobian elliptic function of order N.

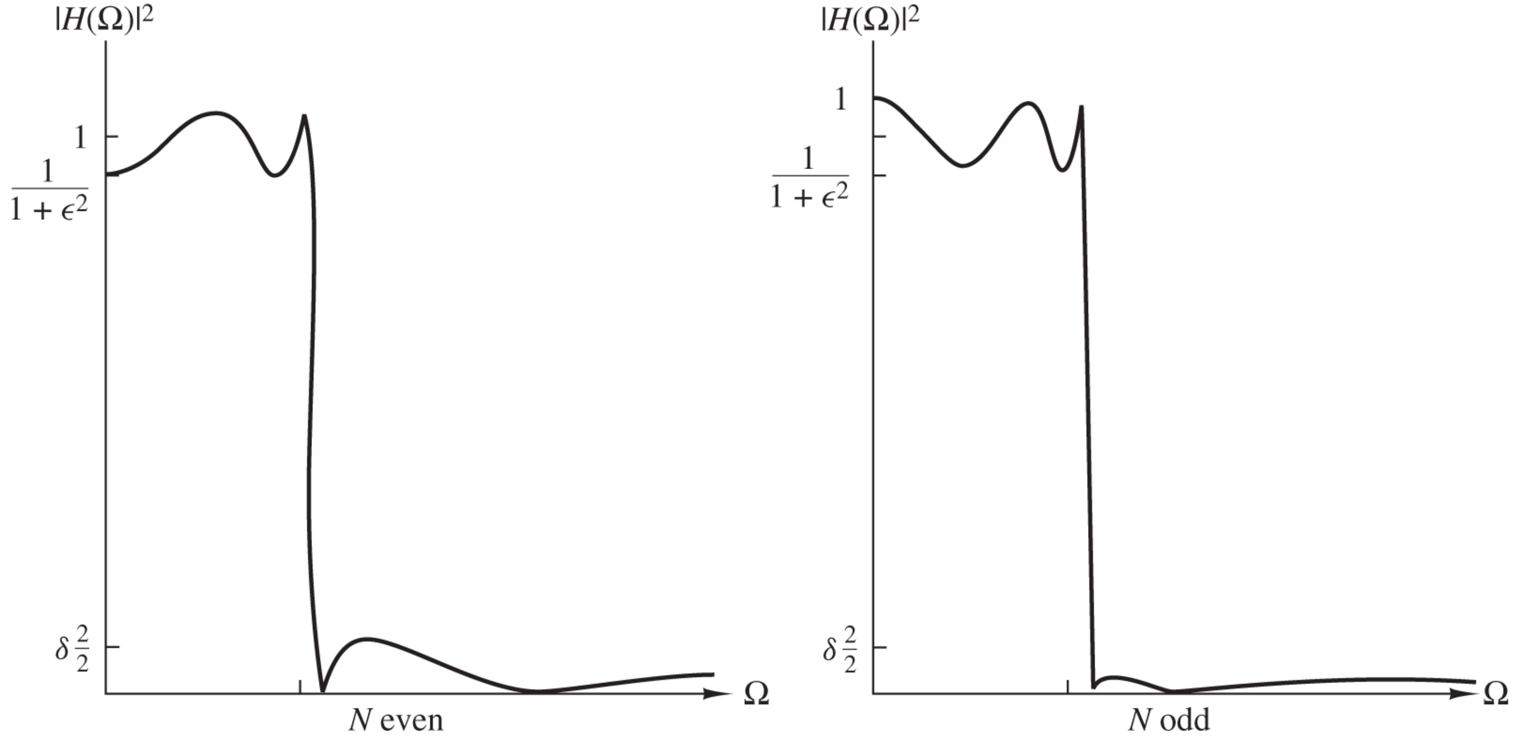


Figure 10.3.14 Magnitude-squared frequency characteristics of elliptic filters.

Bessel Filters

These all-pole filters are monotonic and described by: $H^L(s) = \dfrac{1}{B_N(s)}$

where $B_N(s)$ is the N^{th} order Bessel polynomial

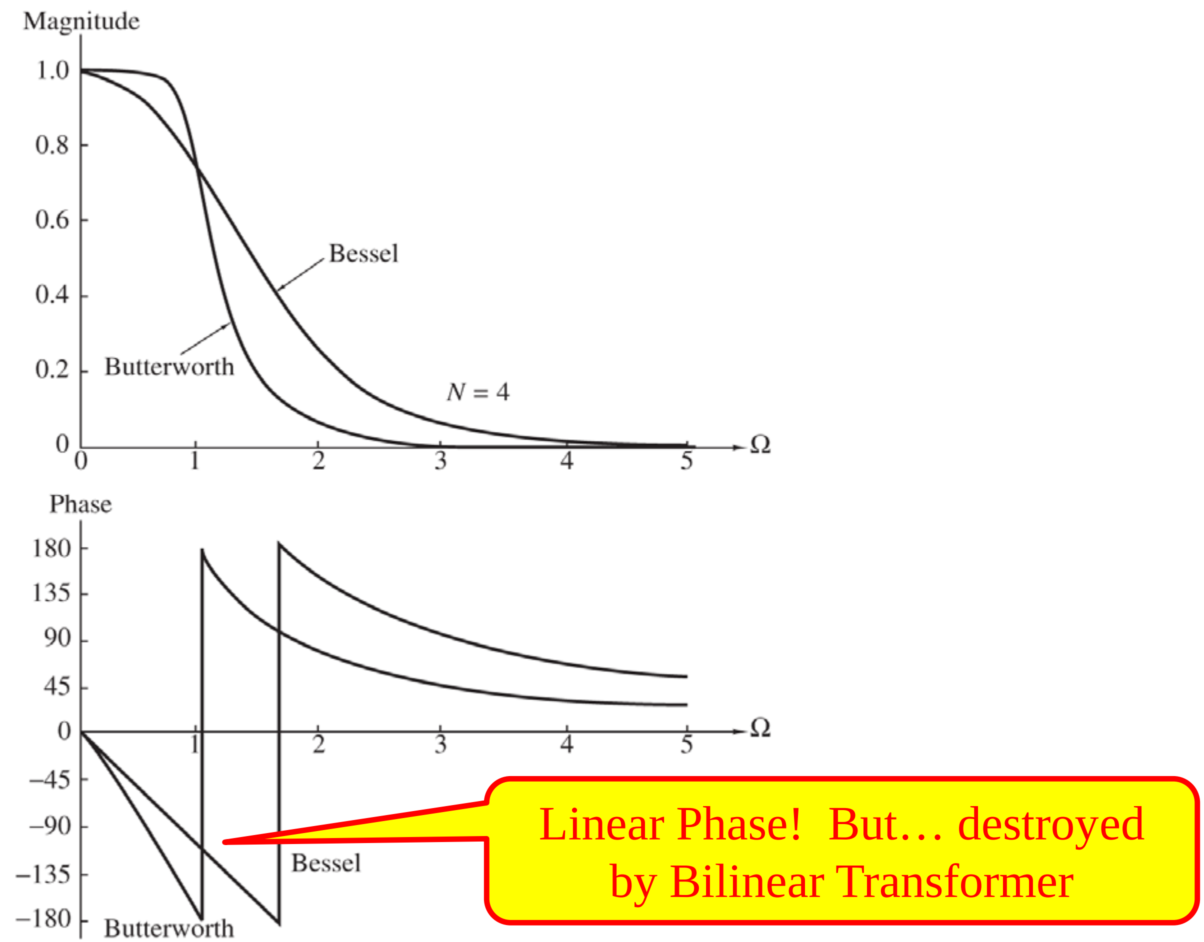


Figure 10.3.15 Magnitude and phase responses of Bessel and Butterworth filters of order $N = 4$.

MATLAB-Based IIR Design

All use classic analog filter results with Bilinear Transform

MATLAB has several easy commands for IIR design:

- butter, cheby1, cheby2, ellip

There is a command for Bessel but its only for analog design

Buttord Butterworth filter order and cutoff frequency

From MATLAB Website

[n,Wn] = buttord(Wp,Ws,Rp,Rs)

Description: Calculates the minimum order of a digital Butterworth filter required to meet a set of filter design specifications.

Returns the lowest order, n, of the digital Butterworth filter with no more than Rp dB of passband "ripple" and at least Rs dB of attenuation in the stopband. The scalar (or vector) of corresponding cutoff frequencies, Wn, is also returned. Use the output arguments n and Wn in butter.

Butter Butterworth filter design

From MATLAB Website

[b,a] = butter(n,Wn)

Description

Returns the transfer function coefficients of an nth-order lowpass digital Butterworth filter with normalized cutoff frequency Wn.

Butterworth IIR Design

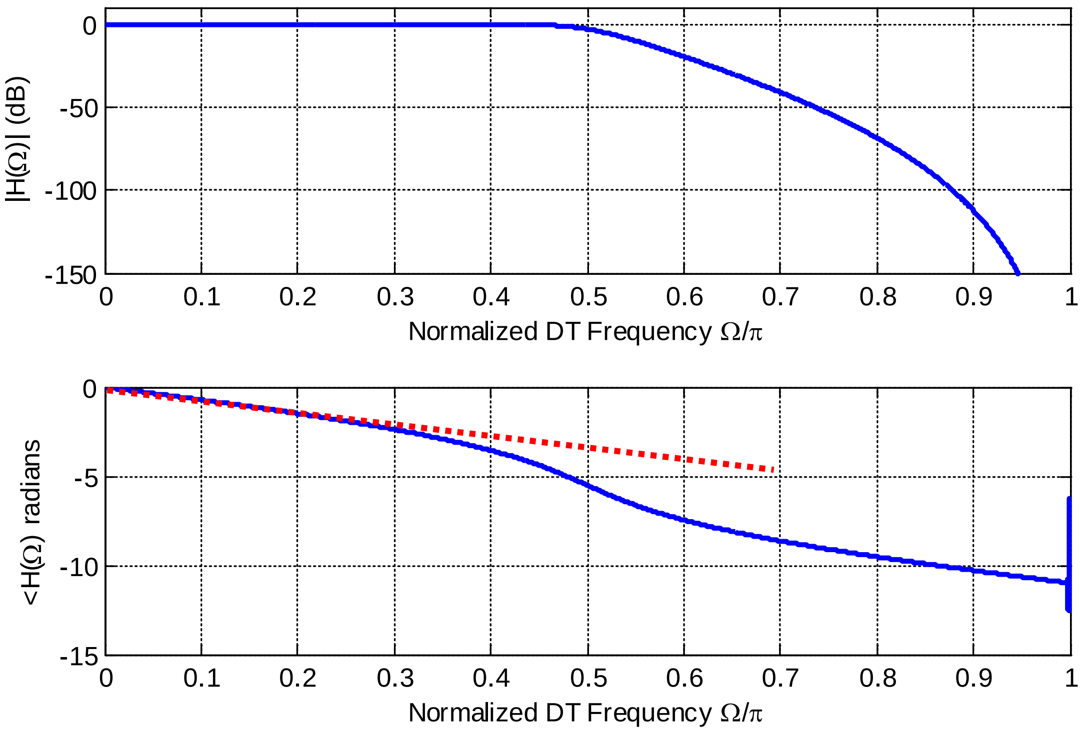
Sets Order

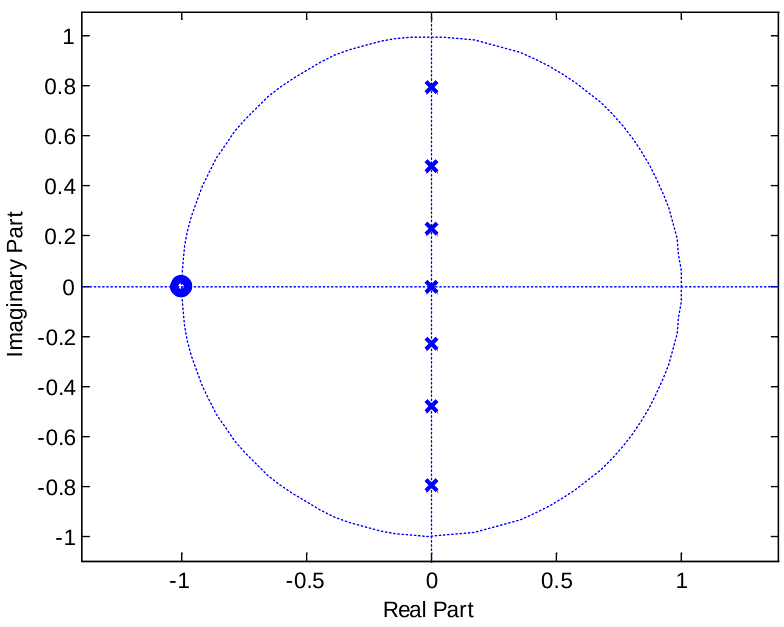
Sets Passband Edge

[b,a] = butter(7,0.5);

b = [0.0166 0.1160 0.3479 0.5798 0.5798 0.3479 0.1160 0.0166]

a = [1.0000 -0.0000 0.9200 -0.0000 0.1927 -0.0000 0.0077 -0.0000]





<u>cheb1ord</u> Chebyshev Type I filter order
[n,Wp] = cheb1ord(Wp,Ws,Rp,Rs)

From MATLAB Website

Description: Calculates the minimum order of a digital Chebyshev Type I filter required to meet a set of filter design specifications.

Returns the lowest order n of the Chebyshev Type I filter that loses no more than Rp dB in the passband and has at least Rs dB of attenuation in the stopband. The scalar (or vector) of corresponding cutoff frequencies Wp, is also returned. Use the output arguments n and Wp with the cheby1 function.

<u>cheby1</u> Chebyshev Type I filter design
[b,a] = cheby1(n,Rp,Wp)

From MATLAB Website

Description: Returns the transfer function coefficients of an nth-order lowpass digital Chebyshev Type I filter with normalized passband edge frequency Wp and Rp decibels of peak-to-peak passband ripple.

Similar commands for Type II filters

Chebyshev (Type II) IIR

[b,a]=cheby2(7,60,0.7);

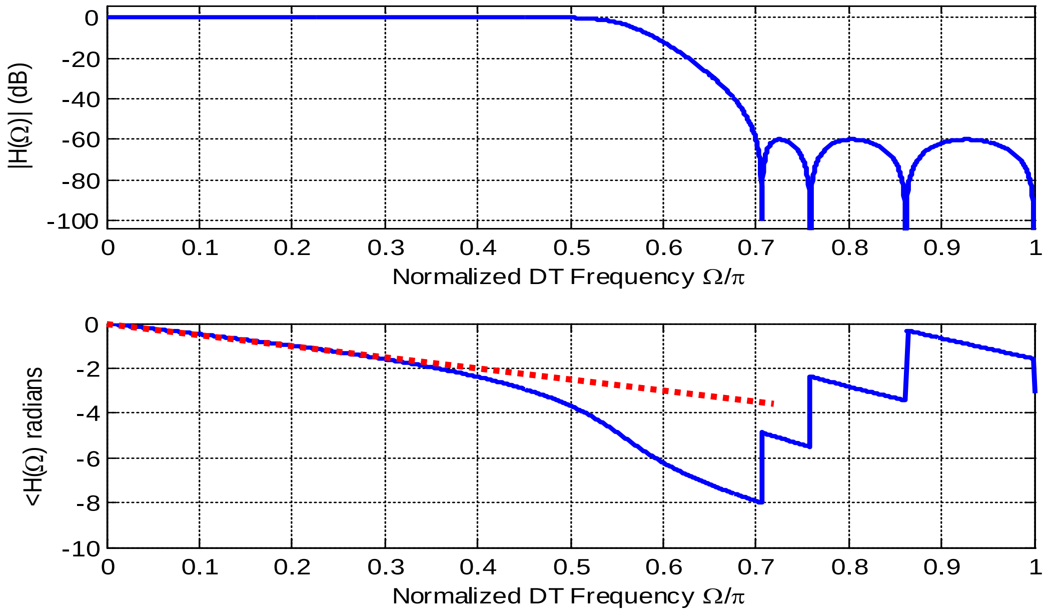
Sets Order

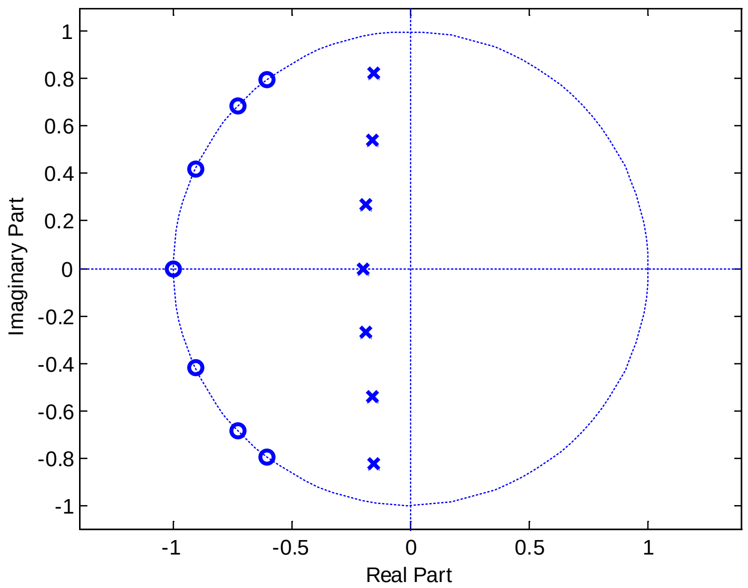
Sets Stopband Height

Sets Stopband Edge

b = [0.0692 0.3789 0.9728 1.5028 1.5028 0.9728 0.3789 0.0692]

a = [1.0000 1.2028 1.6599 1.0991 0.6240 0.2098 0.0473 0.0048]





<u>ellipord</u> Minimum order for elliptic filters

From MATLAB Website

[n,Wp] = ellipord(Wp,Ws,Rp,Rs)

Description: Calculates the minimum order of a digital or analog elliptic filter required to meet a set of filter design specifications.

Returns the lowest order, n, of the elliptic filter that loses no more than Rp dB in the passband and has at least Rs dB of attenuation in the stopband. The scalar (or vector) of corresponding cutoff frequencies Wp, is also returned. Use the output arguments n and Wp in ellip.

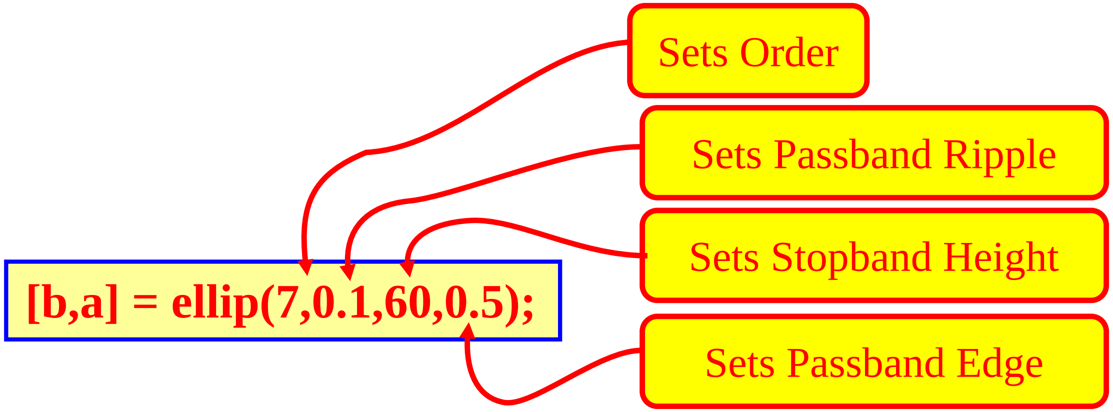
<u>ellip</u> Elliptic filter design

[b,a] = ellip(n,Rp,Rs,Wp)

From MATLAB Website

Description: Returns the transfer function coefficients of an nth-order lowpass digital elliptic filter with normalized passband edge frequency Wp. The resulting filter has Rp decibels of peak-to-peak passband ripple and Rs decibels of stopband attenuation down from the peak passband value.

Elliptical IIR



b = [0.0338 0.1302 0.2821 0.4013 0.4013 0.2821 0.1302 0.0338]

a = [1.0000 -0.8994 2.1386 -1.5364 1.4793 -0.7327 0.3178 -0.0725]