BINGHAMTON
UNIVERSITY
State University of New York

EEO 401
Digital Signal Processing

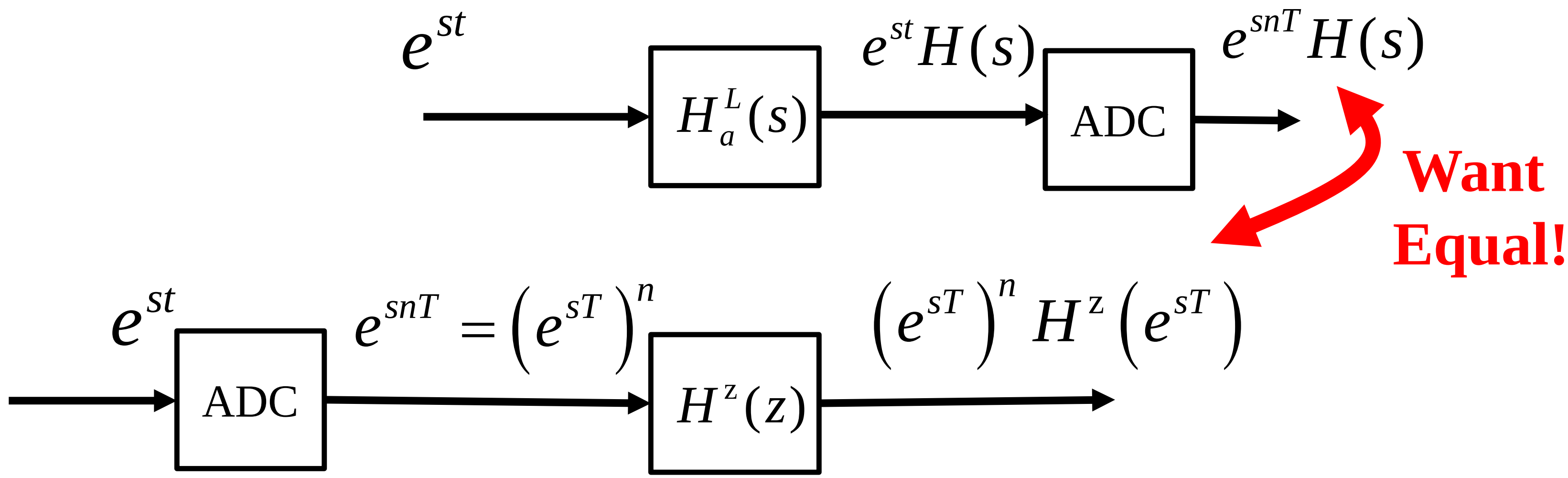
Prof. Mark Fowler

Note Set #33

- IIR Design – Bilinear Transform
- Reading: Sect. 10.3.3 of Proakis & Manolakis

Motivation (different from the P&M textbook)

If the CT and DT systems are equivalent from a transfer function point of view then we should have this kind of equivalence:



Recall that z^n in to DT system gives $H^z(z)z^n$ out

$$H^z\left(e^{sT}\right) = H_a^L(s)$$

Implied map from s to z

$$z = e^{sT}$$

Has two desirable properties:

1. $j\Omega$ axis maps to edge of U.C.
2. LHP maps to inside of U.C.

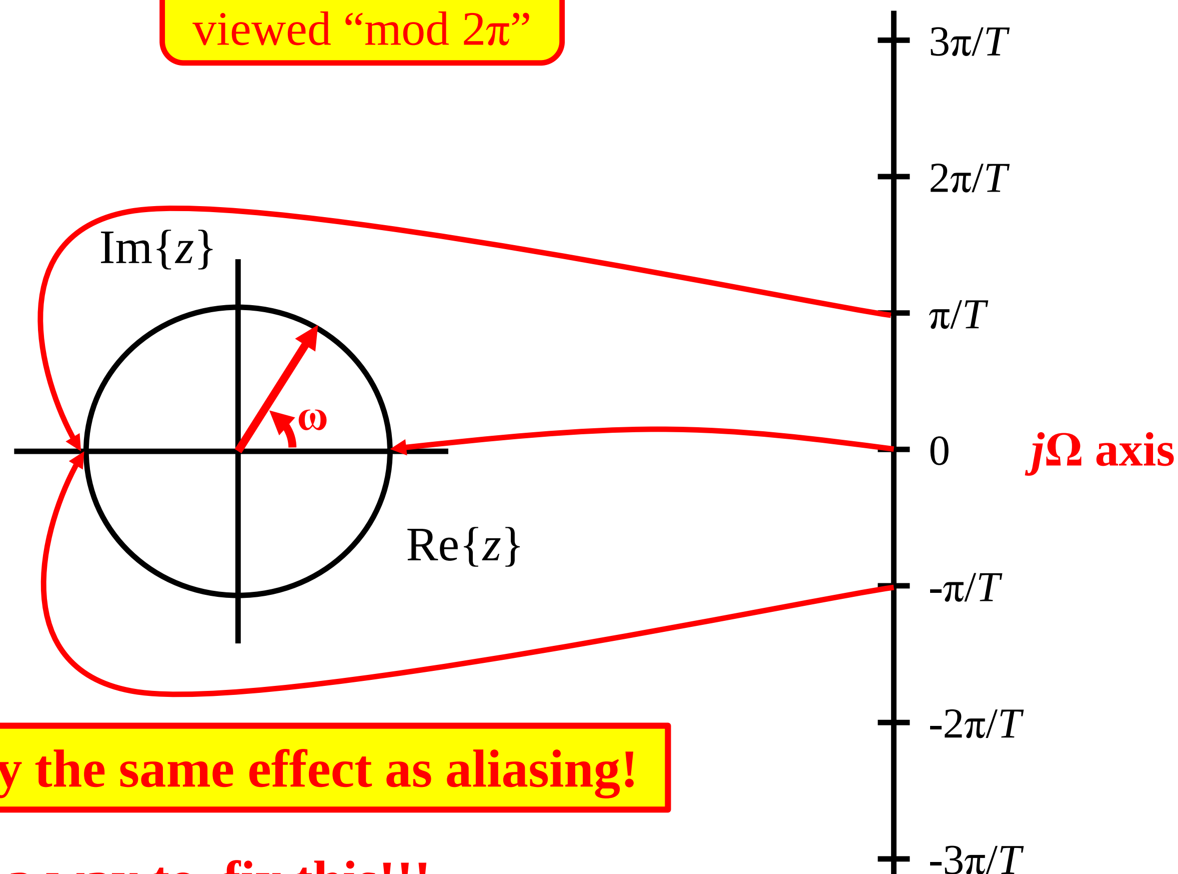
Has two <u>undesirable</u> properties:

1. $s = (1/T)\ln(z)$… resulting $H(z)$ not rational!
2. Mapping wraps the $j\Omega$ axis around & around the U.C.

$$z = e^{sT}\Big|_{s=j\Omega} = e^{j\Omega T}$$

Angle… so viewed "mod 2π"

$$\omega = \Omega T_{\bmod 2\pi}$$



So… this is effectively the same effect as aliasing!

Need to find a way to fix this!!!

Instead of using $z = e^{sT}$… $s = (1/T)\ln(z)$… as the mapping we look for some other mapping that:

- Still maps to the U.C. but without wrapping!!
- And ensures that the resulting $H(z)$ is rational!!

We start with a known expression for the hyperbolic tangent!

$$\tanh\left(\frac{sT}{2}\right) = \frac{e^{sT/2} - e^{-sT/2}}{e^{sT/2} + e^{-sT/2}}$$

And its power series :
$$\tanh\left(\frac{sT}{2}\right) = \frac{sT}{2} - \frac{1}{3}\left(\frac{sT}{2}\right)^3 + \frac{2}{15}\left(\frac{sT}{2}\right)^5 \cdots$$

Equating these two and ignoring these terms (negligible for small T)

$$s \approx \left(\frac{2}{T}\right)\frac{e^{sT/2} - e^{-sT/2}}{e^{sT/2} + e^{-sT/2}} = \left(\frac{2}{T}\right)\frac{e^{sT} - 1}{e^{sT} + 1}$$

2^{nd} step factors out $e^{-sT/2}$ out of top & bottom

From earlier result set to z

Leads to our new map:

Bilinear Transform:
$$s = \left(\frac{2}{T}\right)\frac{z-1}{z+1}$$

Bilinear Transform

$$s = \left(\frac{2}{T}\right)\frac{z-1}{z+1}$$ …OR… $$s = \left(\frac{2}{T}\right)\frac{1-z^{-1}}{1+z^{-1}}$$ $\Rightarrow$ $$H^{f}(z) = H_a^L\left(\frac{2}{T}\frac{z-1}{z+1}\right)$$

Investigate characteristics using $z = re^{j\omega}$ & $s = \sigma + j\Omega$

$$s = \left(\frac{2}{T}\right)\frac{z-1}{z+1} = \left(\frac{2}{T}\right)\frac{re^{j\omega}-1}{re^{j\omega}+1}$$

$$= \underbrace{\left(\frac{2}{T}\right)\frac{r^2-1}{1+r^2+2r\cos(\omega)}}_{=\sigma} + j\underbrace{\left(\frac{2}{T}\right)\frac{2r\sin(\omega)}{1+r^2+2r\cos(\omega)}}_{=\Omega}$$

Note: $r \overset{<}{\underset{>}{=}} 1 \quad \Rightarrow \quad \sigma \overset{<}{\underset{>}{=}} 0$ $\Rightarrow$

… stability is preserved!!
… $j\Omega$ axis maps to u.c. edge!!

Need to see if we've fixed the "wrapped around & around" problem…

Bilinear Transform: Mapping between CT & DT Frequencies

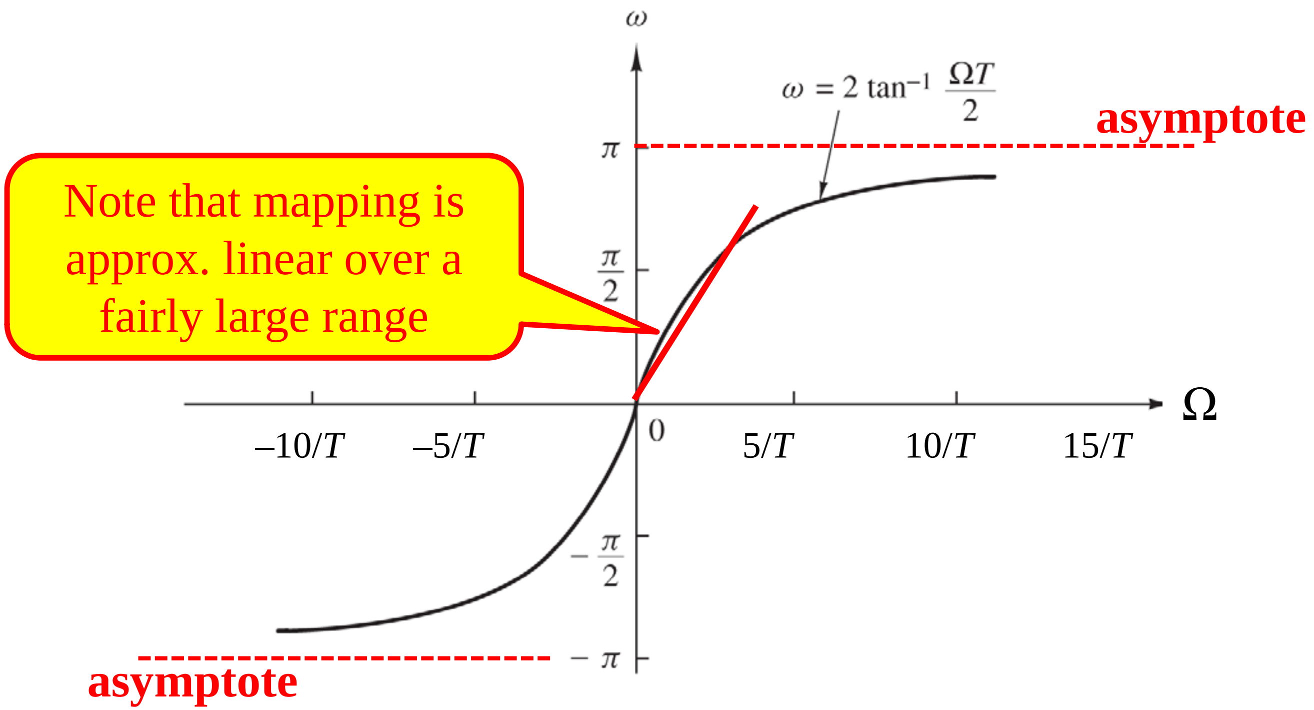
Setting $r = 1$ gives $\sigma = 0$ so that is the condition for mapping between CT & DT frequencies:

$$\Omega = \left(\frac{2}{T}\right)\frac{2r\sin(\omega)}{1+r^2+2r\cos(\omega)}\bigg|_{r=1} = \left(\frac{2}{T}\right)\frac{\sin(\omega)}{1+\cos(\omega)} = \left(\frac{2}{T}\right)\tan\left(\frac{\omega}{2}\right)$$

Or…

$$\omega = 2\tan^{-1}\left(\frac{\Omega T}{2}\right)$$

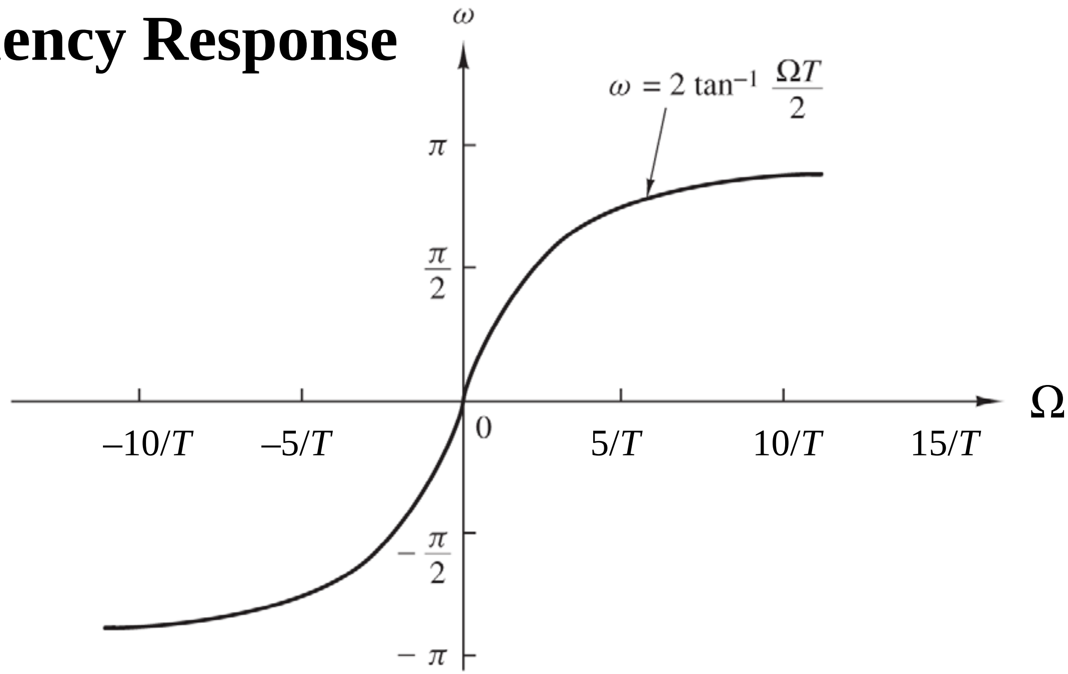
which is quite different from before: $\omega = \Omega T_{\text{mod } 2\pi}$

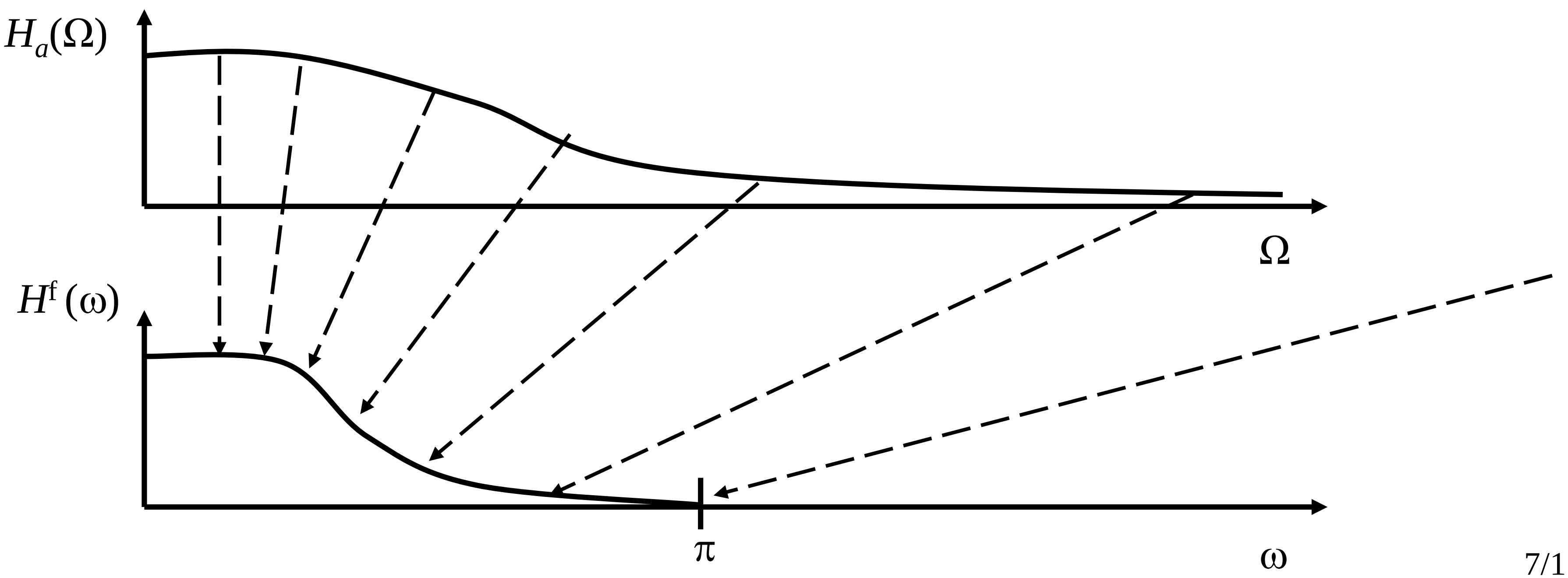


So… this mapping does not wrap… But it "warps"!!!

Effect of Warping on Frequency Response

$$\omega = 2\tan^{-1}\left(\frac{\Omega T}{2}\right)$$





Matlab Command for Bilinear Transform

[NUMd,DENd] = bilinear(NUM,DEN,Fs),

$F_s = 1/T$

where NUM and DEN are row vectors containing numerator and denominator transfer function coefficients, NUM(s)/DEN(s), in descending powers of s, transforms to z-transform coefficients NUMd(z)/DENd(z).

Choosing Value of T

Two different approaches:

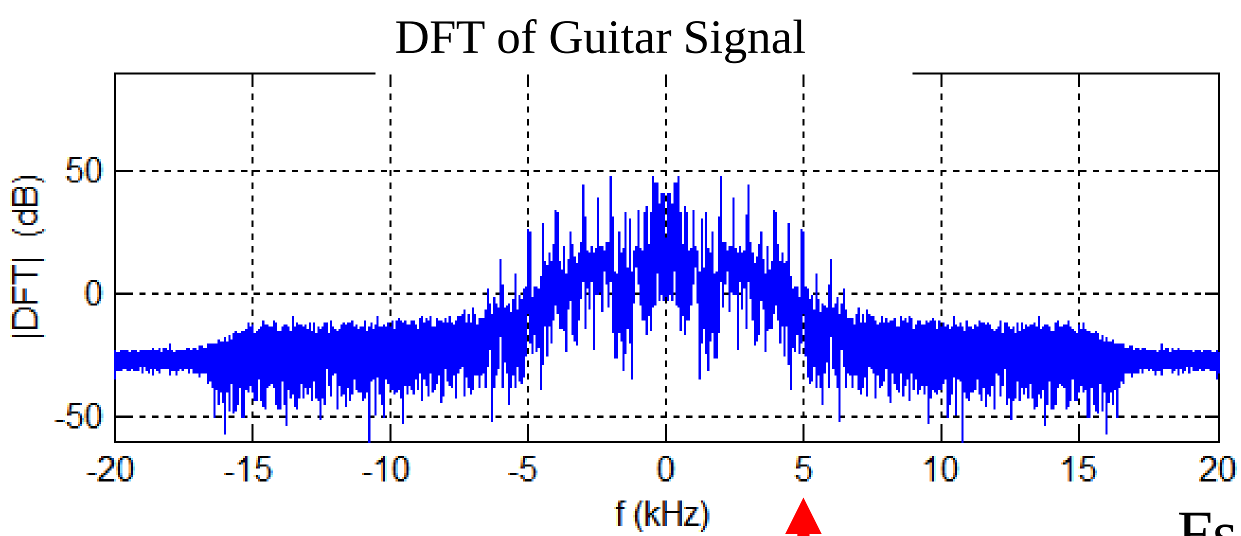
1. Pick *T* for convenience in the filter design
   - This requires picking a different sampling rate for the signal that matches its bandwidth
   - This approach requires determining frequency cutoffs in the digital domain and mapping them to some CT filter
2. Pick *T* such that $F_s = 1/T$ is appropriate for the signal sampling
   - Then deal with its impact in the filter design
   - This approach requires an analog filter design with cutoff frequencies spec'd according to the signal's CT requirements

Example of Choosing T via Method #1

Based on Example 10.3.4 in textbook

1. Pick T for convenience in the filter design
    - This requires picking a different sampling rate for the signal that matches its bandwidth
    - This approach requires determining frequency cutoffs in the digital domain and mapping them to some CT filter

- Suppose we have a signal that is appropriately sampled at some F_s.
- With that sampling… suppose we determine we want an IIR filter with a resonant peak at DT frequency $\omega = \pi/4$ (which corresponds to CT $f = F_s/8$)



Fs/2 = 20kHz

Want peak here!

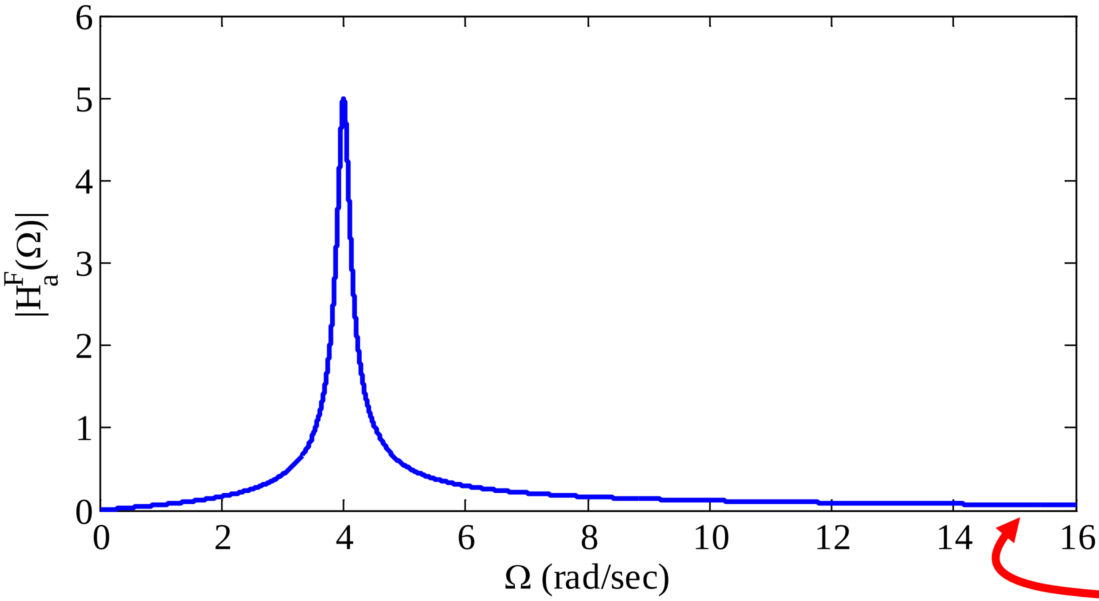
Now we pick an analog filter with resonance… and it does not have to be designed to have a peak at 5kHz!!! We pick the *T* as needed to map THIS filter's peak frequency to DT frequency of π/4 rad/sample

$$H_a^L(s) = \frac{s+0.1}{(s+0.1)^2+16} = \frac{s+0.1}{s^2+0.2s+16.01} = \frac{s+0.1}{\left(s+0.1+j4\right)\left(s+0.1-j4\right)}$$

This has a resonant peak at Ω = 4 rad/sec

Far cry from the "real" 2πx5000

```
>> w=0:0.001:16;  H=freqs([1 0.1],[1 0.2 16.01],w);    plot(w,abs(H))
```



Want to map Ω = 4 to ω = π/4

$$\Omega = \left(\frac{2}{T}\right)\tan\left(\frac{\omega}{2}\right)$$

$$4 = \left(\frac{2}{T}\right)\tan\left(\frac{\pi/4}{2}\right)$$

$$T = \frac{\tan\left(\frac{\pi/4}{2}\right)}{2} = 0.2071$$

Satisfactory location of "Folding Freq"

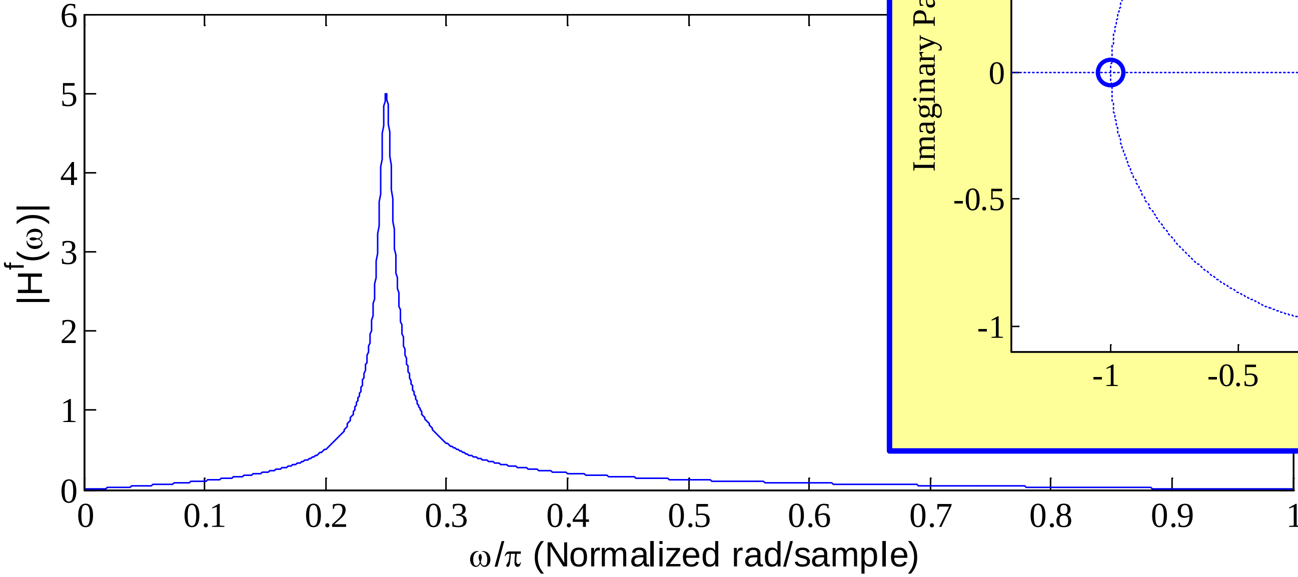
Now use bilinear transform:

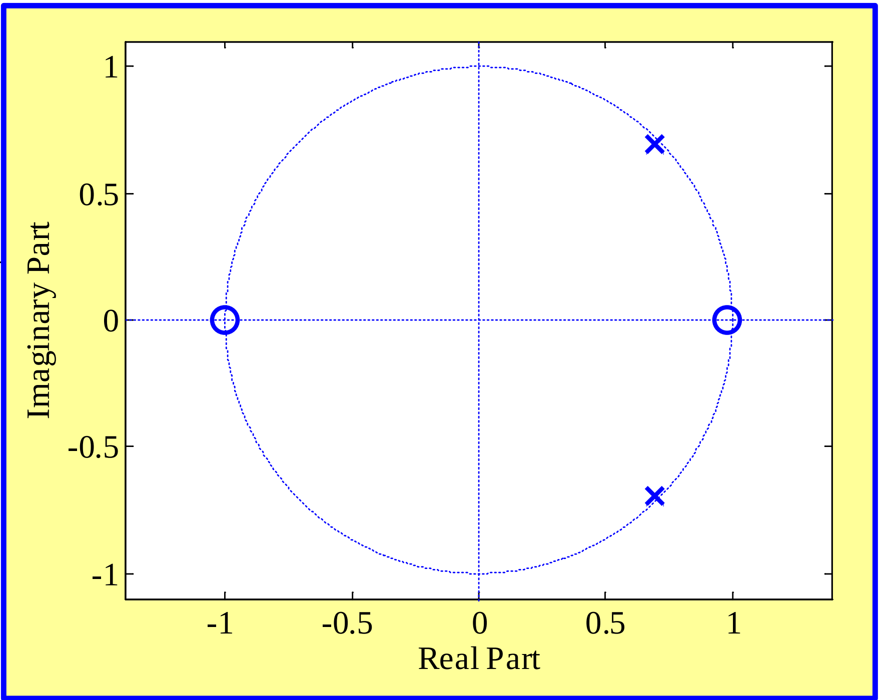
1/T — From filter needs, not signal needs!

```
>> [NUMd,DENd] = bilinear([1 0.1],[1 0.2 16.01],1/0.2071)
NUMd =
    0.0877    0.0018   -0.0859
DENd =
    1.0000   -1.3894    0.9653
```

$$H_a^L(s) = \frac{s+0.1}{s^2+0.2s+16.01}$$

$$H^z(z) = \frac{0.0877+0.0018z^{-1}-0.0859z^{-2}}{1-1.3894z^{-1}+0.9653z^{-2}}$$



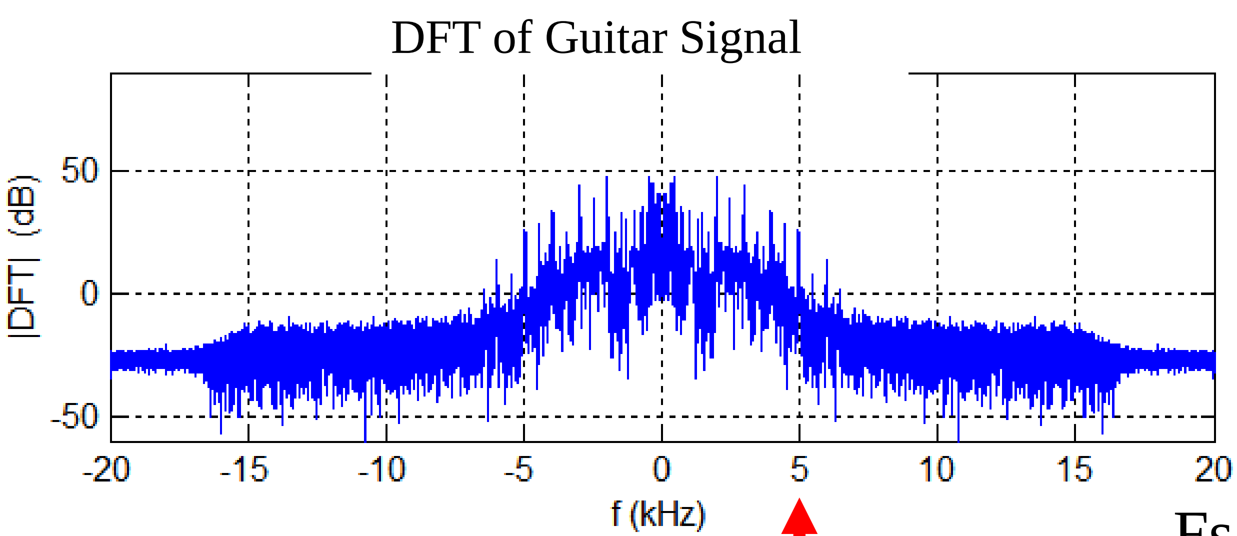


Example of Choosing *T* via Method #2

2. Pick T such that Fs = 1/T is appropriate for the signal sampling
   - Then deal with impact in the filter design
   - This approach requires an analog filter design with cutoff frequencies spec'd according to the signal's CT requirements

- Suppose we have a signal like that shown below and we want to lowpass filter it with a cutoff of 5kHz
- Fs = 40,000 is an appropriate sampling rate

$$T = 1/F_s = 1/40,000 = 2.5 \times 10^{-5}$$



Fs/2 = 20kHz

Want cutoff here!

Suppose we choose as our analog prototype a first-order LPF:

$$H_a^L(s) = \frac{\Omega_c}{s + \Omega_c}$$

Ω_c is the cutoff on the analog frequency axis

However, we can **not** use Ω_c = 2πx5000 = 31,415.93 rad/sec because of the warping of the bilinear transform!!

Signal frequencies get mapped linearly according to F_s:

$$5000 \text{ Hz} \quad \rightarrow \quad 5000 \times (2\pi/F_s) = 5000 \times (2\pi/40000) = \pi/4 \text{ rad/sample}$$

But… CT filter frequencies get mapped by the bilinear's warp: $\omega = 2\tan^{-1}\left(\frac{\Omega T}{2}\right)$

So we "***pre-warp***" the 5000 Hz requirement using the inverse of this

$$\Omega = \left(\frac{2}{T}\right)\tan\left(\frac{\omega}{2}\right) \Rightarrow \Omega_c = \left(\frac{2}{T}\right)\tan\left(\frac{5000 \times (2\pi/F_s)}{2}\right) = 2F_s \tan\left(\frac{5000 \times (2\pi/F_s)}{2}\right)$$

$$\Rightarrow \Omega_c = 33{,}137.08 \text{ rad/sec}$$

"Pre-warped" higher than this

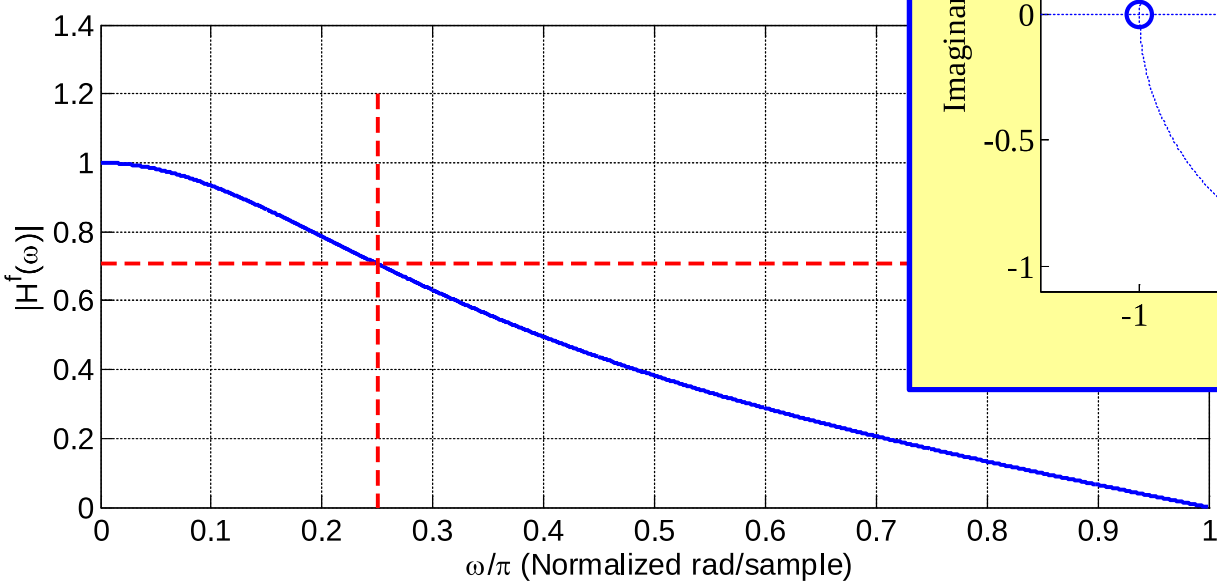
Now use bilinear transform:

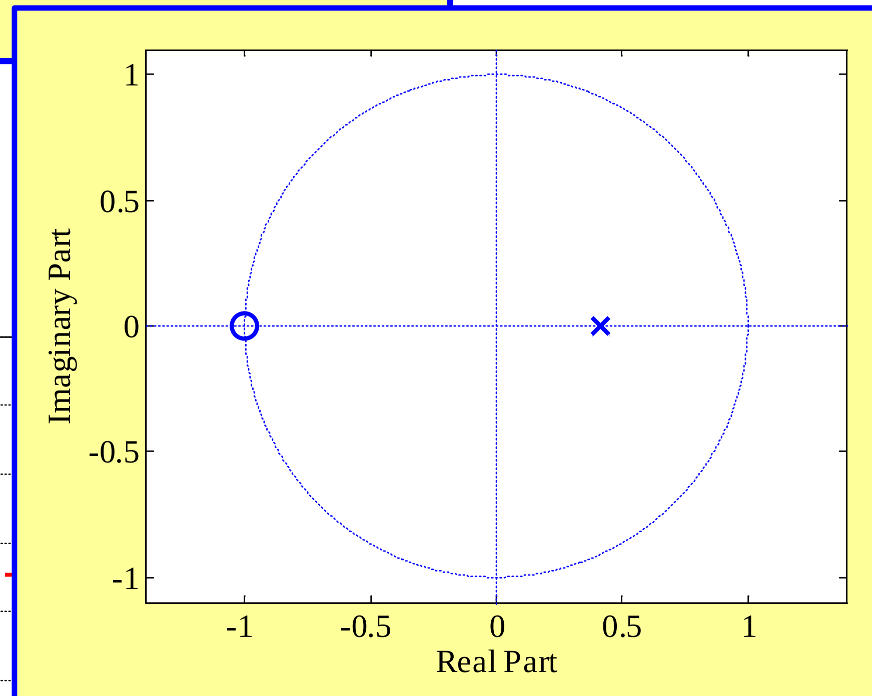
F_s From signal needs!

```
>> [NUMd,DENd] = bilinear([33137.08],[1 33137.08],40000)
NUMd =
   0.2929    0.2929
DENd =
   1.0000   -0.4142
```

$$H_a^L(s) = \frac{33{,}137.08}{s + 33{,}137.08}$$

$$H^z(z) = \frac{0.2929 + 0.2929 z^{-1}}{1 - 0.4142 z^{-1}}$$





Using "pre-warp" feature of bilinear.m command

[NUMd,DENd] = bilinear(NUM,DEN,Fs,Fp),

applies prewarping before the bilinear transformation so that the frequency responses before and after mapping match exactly at frequency point Fp (match point Fp is **specified in Hz**).

where NUM and DEN are row vectors containing numerator and denominator transfer function coefficients, NUM(s)/DEN(s), in descending powers of s, transforms to z-transform coefficients NUMd(z)/DENd(z).

```
>> [NUMd,DENd] = bilinear([33137.08],[1 33137.08],40000)

NUMd =

   0.2929    0.2929

DENd =

   1.0000   -0.4142
```

SAME!

```
>> [NUMd,DENd] = bilinear([31415.93],[1 31415.93],40000,31415.93/(2*pi))
NUMd =

   0.2929    0.2929

DENd =

   1.0000   -0.4142
```

$\Omega_c = 2\pi x5000 = 31{,}415.93$ rad/sec