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# EEO 401
# Digital Signal Processing
## Prof. Mark Fowler

**Note Set #31**

- Linear Phase FIR Design – Optimum Equiripple (Parks-McClellan)
- Reading: Sect. 10.2.4 – 10.2.6 of Proakis & Manolakis

# Motivation

The window method and the frequency sampling method have a major drawback:

- Can't precisely control $\omega_p$ & $\omega_s$ and $\delta_p$ & $\delta_s$

**Require iterative tweaking to get desired specs**

The **optimal equiripple method** allows easy numerical design that can meet specs on band edges and band ripples with essentially the lowest order FIR filter!

Define $H_{dr}(\omega)$ — *Amplitude* of **desired** filter frequency response

$H_{r,\{b[n]\}}(\omega)$ — *Amplitude* of **designed** filter frequency response

Coefficients of design

Weight… can emphasize certain band

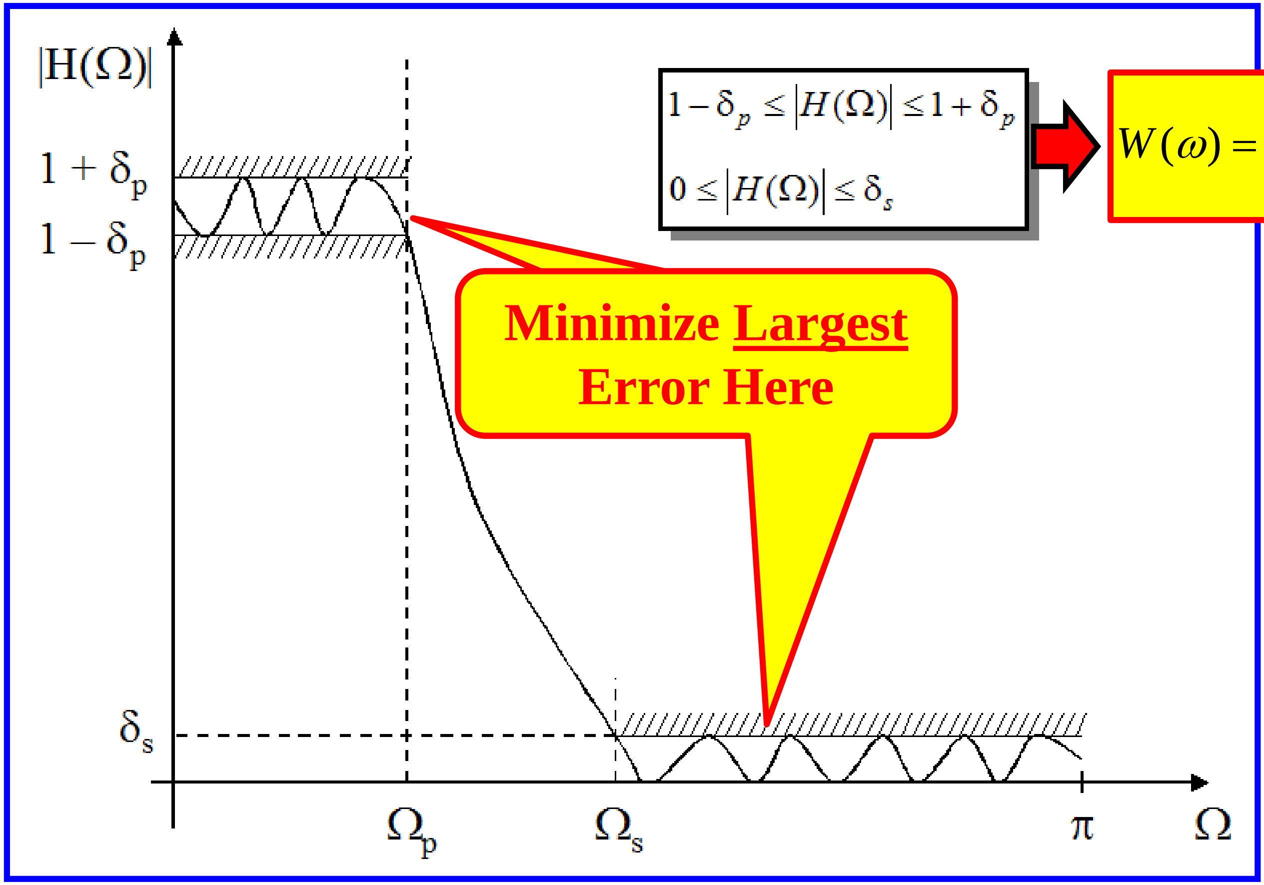
Define Weighted Error: $E_{\{b[n]\}}(\omega) = W(\omega)\left[H_{dr}(\omega) - H_{r,\{b[n]\}}(\omega)\right]$

Design to satisfy: $\boxed{\min_{\text{over } \{b[n]\}} \left[ \max_{\omega \in PB \cup SB} \left| E_{\{b[n]\}}(\omega) \right| \right]}$

**Minimize Largest Error**

Passbands & Stopbands

$$\min_{\text{over } \{b[n]\}} \left[ \max_{\omega \in PB \cup SB} \left| W(\omega) \left[ H_{dr}(\omega) - H_{r,\{b[n]\}}(\omega) \right] \right| \right]$$




$$1 - \delta_p \le |H(\Omega)| \le 1 + \delta_p$$

$$0 \le |H(\Omega)| \le \delta_s$$

$$W(\omega) = \begin{cases} \delta_s / \delta_p, & \omega \in \text{passband} \\ 1, & \omega \in \text{stopband} \end{cases}$$

# <u>Design Basis</u>

This approach was proposed and solved by Parks & McLellan in 1972…

… <u>three clever math "tricks"</u> were used to develop a design algorithm

1. Find a <u>single common form</u> for all four linear phase cases
   - Enables a single algorithm that works for all four cases
2. Exploit a <u>theorem</u> from "Chebyshev" approximation theory
   - "<u>Alternation Theorem</u>": gives an easy to apply necessary and sufficient condition for optimal Chebyshev approximation
   - Specifies how many "extremal error" frequencies there must be and that the error alternates sign over these frequencies
3. Exploit an iterative <u>algorithm</u> from "Chebyshev" approximation theory
   - "<u>Remez Exchange Algorithm</u>": given the current "guess" of the extremal frequencies… it updates to better ones… converges to solution.

We'll discuss trick #1 here because it provides the FIR designer insight into characteristics important to the designer.

The other two we do not discuss… they are details only needed if you are implementing the design algorithm (which is available in MATLAB).

# Common Form for Four Linear Phase FIR Cases

| Symmetry \ Length | *M* Odd | *M* Even |
| --- | --- | --- |
| **Symmetric** | Case #1 | Case #2 |
| **Antisymmetric** | Case #3 | Case #4 |

## Case #1: Symmetric Impulse Response & Lengh *M* Odd

$$\boxed{h[n] = h[M-1-n]}$$

For this case the book shows (sect 10.2.1) that

$$H_r(\omega) = h\big[(M-1)/2\big] + 2\sum_{n=0}^{(M-3)/2} h[n]\cos\Big(\omega\big(\tfrac{M-1}{2} - n\big)\Big)$$

Let $k = (M-1)/2 - n$ and define

$$a[k] = \begin{cases} h\big[(M-1)/2\big], & k = 0 \\ 2h\big[(M-1)/2 - k\big], & k = 1, 2, \ldots, (M-1)/2 \end{cases}$$

$$\Rightarrow \quad \boxed{H_r(\omega) = \sum_{k=0}^{(M-1)/2} a[k]\cos(\omega k)}$$

# Case #2: Symmetric Impulse Response & Lengh *M* Even

$$\boxed{h[n] = h[M-1-n]}$$

For this case the book shows (sect 10.2.1) that

$$H_r(\omega) = 2\sum_{n=0}^{(M-1)/2} h[n]\cos\left(\omega\left(\tfrac{M-1}{2} - n\right)\right)$$

Let $k = M/2 - n$ and define $b[k] = 2h\left[M/2 - k\right], \quad k = 1, 2, \ldots, (M-1)/2$

$$\Rightarrow \quad H_r(\omega) = \sum_{k=1}^{M/2} b[k]\cos\left(\omega(k - 1/2)\right)$$

Define $\tilde{b}[0] = \tfrac{1}{2} b[1]$

$$\tilde{b}[k] = 2b[k] - \tilde{b}[k-1], \quad k = 2, 3, \ldots, (M/2) - 2$$

$$\tilde{b}[(M/2) - 1] = 2b[M/2]$$

$$\Rightarrow \quad \boxed{H_r(\omega) = \cos\left(\omega/2\right) \sum_{k=0}^{(M/2)-1} \tilde{b}[k]\cos\left(\omega k\right)}$$

# Case #3: Anti-Symmetric Impulse Response & Lengh *M* Odd

$\boxed{h[n] = -h[M-1-n]}$ For this case the book shows (sect 10.2.1) that

$$H_r(\omega) = 2\sum_{n=0}^{(M-3)/2} h[n]\sin\left(\omega\left(\tfrac{M-1}{2} - n\right)\right)$$

Let $k = (M-1)/2 - n$ and define $c[k] = 2h\big[(M-1)/2 - k\big], \quad k = 1, 2, \ldots, (M-1)/2$

$$\Rightarrow \quad H_r(\omega) = \sum_{k=1}^{(M-1)/2} c[k]\sin(\omega k)$$

$$\Rightarrow \quad \boxed{H_r(\omega) = \sin(\omega)\sum_{k=0}^{(M-3)/2} \tilde{c}[k]\cos(\omega k)}$$

See Eq. (10.2.57) in textbook

# Case #4: Anti-Symmetric Impulse Response & Lengh *M* Even

$\boxed{h[n] = -h[M-1-n]}$ For this case the book shows (sect 10.2.1) that

$$H_r(\omega) = 2\sum_{n=0}^{(M/2)-1} h[n]\sin\left(\omega\left(\tfrac{M-1}{2} - n\right)\right)$$

Let $k = M/2 - n$ and define $d[k] = 2h\left[M/2 - k\right], \quad k = 1, 2, \ldots, M/2$

$$\Rightarrow \quad H_r(\omega) = \sum_{k=1}^{M/2} d[k]\sin\left(\omega(k - \tfrac{1}{2})\right)$$

$$\Rightarrow \quad \boxed{H_r(\omega) = \sin\left(\omega/2\right)\sum_{k=0}^{(M/2)-1} \tilde{d}[k]\cos\left(\omega k\right)}$$

See Eq. (10.2.62) in textbook

# Compare all the forms:

**Case #1** $H_r(\omega) = 1 \sum_{k=0}^{(M-1)/2} a[k]\cos(\omega k)$

**Case #2** $H_r(\omega) = \cos(\omega/2) \sum_{k=0}^{(M/2)-1} \tilde{b}[k]\cos(\omega k)$

**Case #3** $H_r(\omega) = \sin(\omega) \sum_{k=0}^{(M-3)/2} \tilde{c}[k]\cos(\omega k)$

**Case #4** $H_r(\omega) = \sin(\omega/2) \sum_{k=0}^{(M/2)-1} \tilde{d}[k]\cos(\omega k)$

$$\boxed{H_r(\omega) = Q(\omega)P(\omega)}$$

**Common Form!**

$$\boxed{P(\omega) = \sum_{k=0}^{L} \alpha[k]\cos(\omega k)}$$

$$\boxed{Q(\omega) = \begin{cases} 1, & \text{Case \#1} \\ \cos(\omega/2), & \text{Case \#2} \\ \sin(\omega), & \text{Case \#3} \\ \sin(\omega/2), & \text{Case \#4} \end{cases}}$$

Now… revisit the Error Equation:

$$E_{\{b[n]\}}(\omega) = W(\omega)\left[H_{dr}(\omega) - H_{r,\{b[n]\}}(\omega)\right]$$

And… plug common form into the Error Equation:

$$\begin{aligned} E_{\{\alpha[n]\}}(\omega) &= W(\omega)\left[H_{dr}(\omega) - Q(\omega)P_{\{\alpha[n]\}}(\omega)\right] \\ &= W(\omega)Q(\omega)\left[\frac{H_{dr}(\omega)}{Q(\omega)} - P_{\{\alpha[n]\}}(\omega)\right] \end{aligned}$$

Since $Q(\omega)$ does not depend on the filter coefficients:

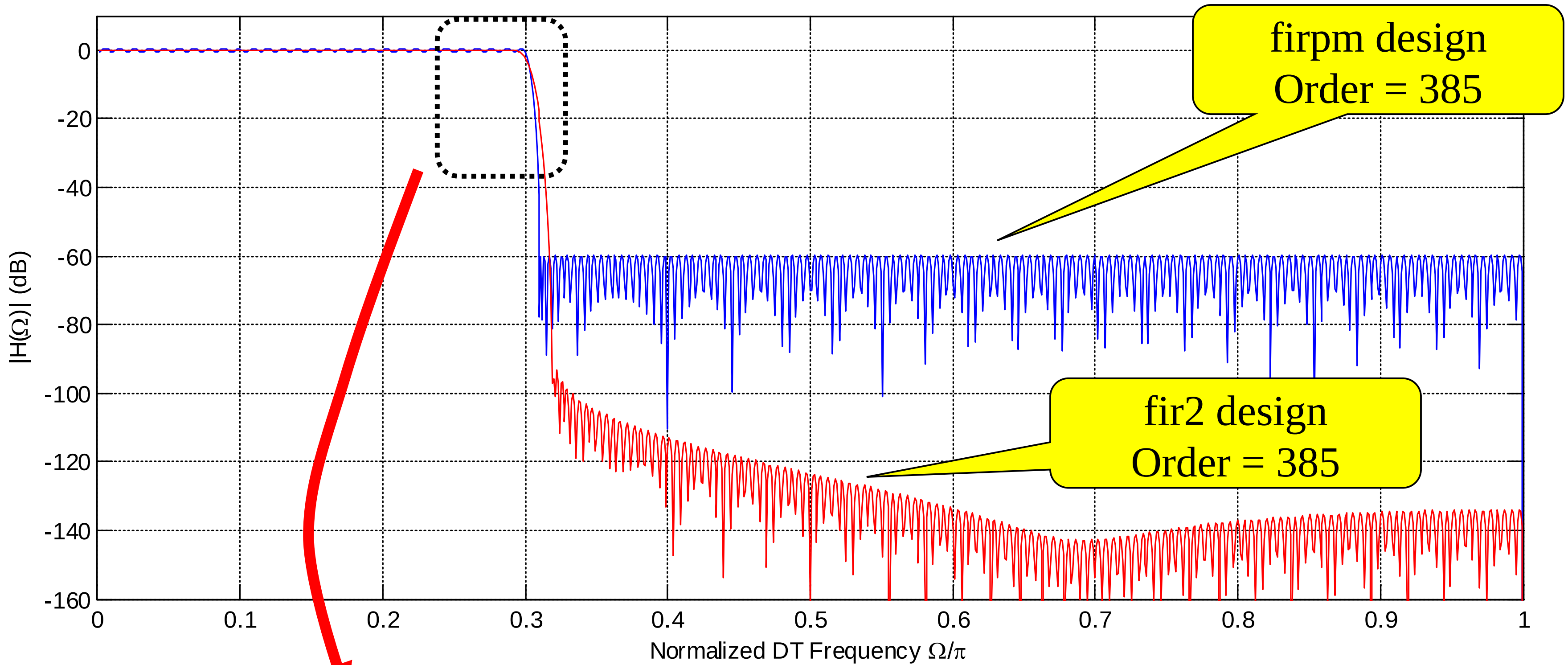
$$E_{\{\alpha[n]\}}(\omega) = \hat{W}(\omega)\left[\hat{H}_{dr}(\omega) - P_{\{\alpha[n]\}}(\omega)\right]$$

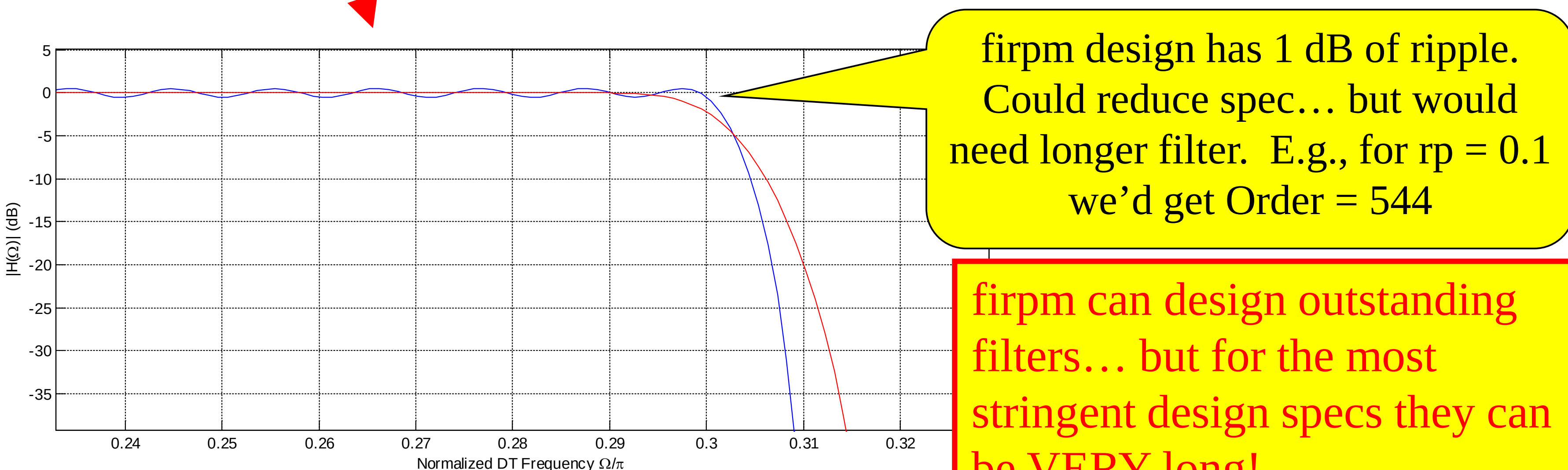
**Contains Q(ω) based on specific case**

**Common Form!**

```
% Lowpass Filter Design Specifications:
%           Passband cutoff frequency = 0.3π rad/sample
%           Stopband cutoff frequency = 0.31π rad/sample
% ·         At least 60 dB of stopband attenuation
% ·         No more than 1 dB passband ripple

rp=1; rs=60; % specify passband ripple  & stopband attenuation in dB
f_spec=[0.3 0.31];   % specify passband and stopband edges in normalized DT freq
AA=[1 0];    %%%  specfies that you want a lowpass filter
dev=[(10^(rp/20)-1)/(10^(rp/20)+1) 10^(-rs/20)];   % parm. needed by design routine
Fs=2;   % “Fake” value for Fs so our design is done in terms of normalized DT freq

[N,fo,ao,w]=firpmord(f_spec,AA,dev,Fs);
% estimates filter order and gives other parms needed to run firpm

b=firpm(N,fo,ao,w);     % Computes the designed filter coefficients in vector b
```

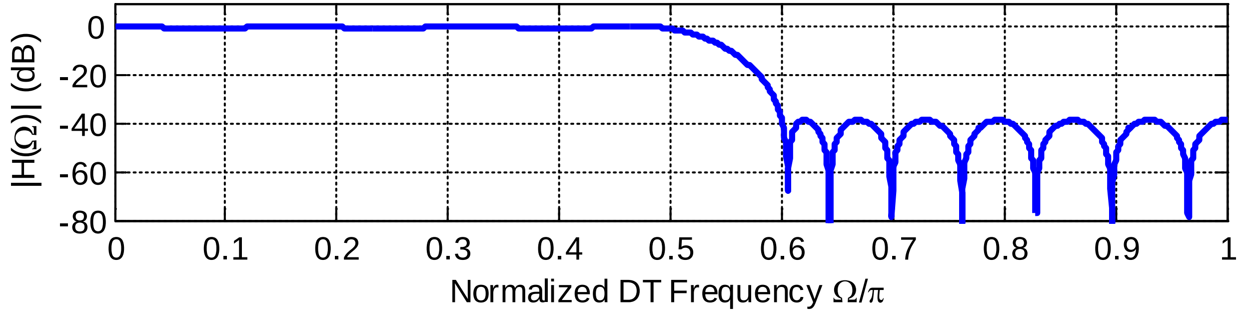
The resulting value for the order for this design is 385!!

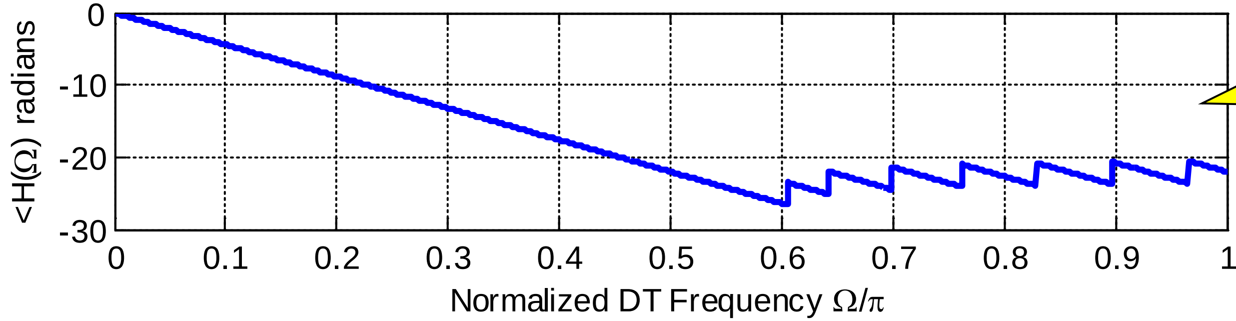



firpm can design outstanding filters… but for the most stringent design specs they can be VERY long!

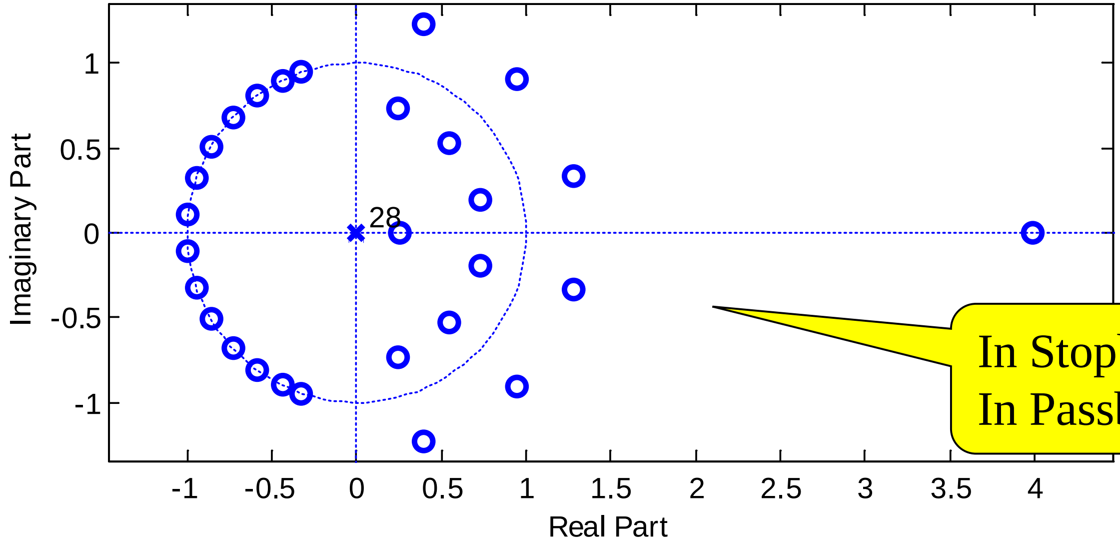
Let's look at pole-zero plot for a simpler firpm-designed filter…




**Linear Phase… all designs by firpm have this very desirable trait!!!**

>> zplane(b,1)

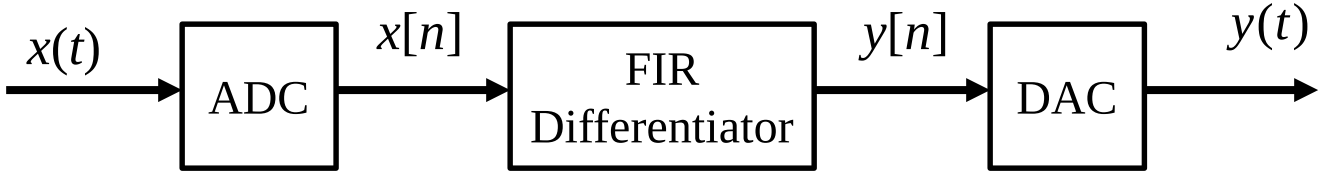



In Stopband: zeros placed right on UC
In Passband: zeros "line" the UC

# Design of Equiripple FIR Differentiators

A particular filter that is sometimes needed is a filter that computes samples of the derivative of the underlying CT signal whose samples you have.

**Desired!**




$$y(t) = \frac{dx(t)}{dt}$$

From properties of Laplace Transform for CT signals we know that a derivative in the time domain corresponds to multiplication by *s* in the *s*-domain

$$H^L(s) = s \qquad\qquad H^F(\omega) = j\omega$$

**CT rad/sec**

So… trying to mimic this with a DT filter we have

$$H_d^{\mathrm{f}}(\omega) = j\omega, \quad -\pi \le \omega \le \pi \quad (2\pi\text{-periodic})$$

**Desired**

## To explore this… compute its impulse response

$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d^{\mathrm{f}}(\omega) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} j\omega e^{j\omega n} d\omega$$

$$= \frac{\cos(\pi n)}{n}$$

**Antisymmetric: $h_d[-n] = -h_d[n]$**

Thus… use Case #3 (*M* odd) or Case #4 (*M* even) linear phase FIR filters

$$h[n] = -h[M-1-n]$$

Both cases satisfy $H_r(0) = 0$ as needed for the differentiator

However, for *M* odd we have $H_r(\pi) = 0$, which is not allowable for a “true” differentiator. But… in practice we often only need to meet the desired response over a limited range:

$$H_d^{\mathrm{f}}(\omega) = \begin{cases} j\omega, & -\omega_p \le \omega \le \omega_p \\ 0, & |\omega| \le \omega_p \end{cases} \quad (2\pi\text{-periodic})$$

***M* Odd or Even OK**

# Design of Equiripple Differentiators via firpm

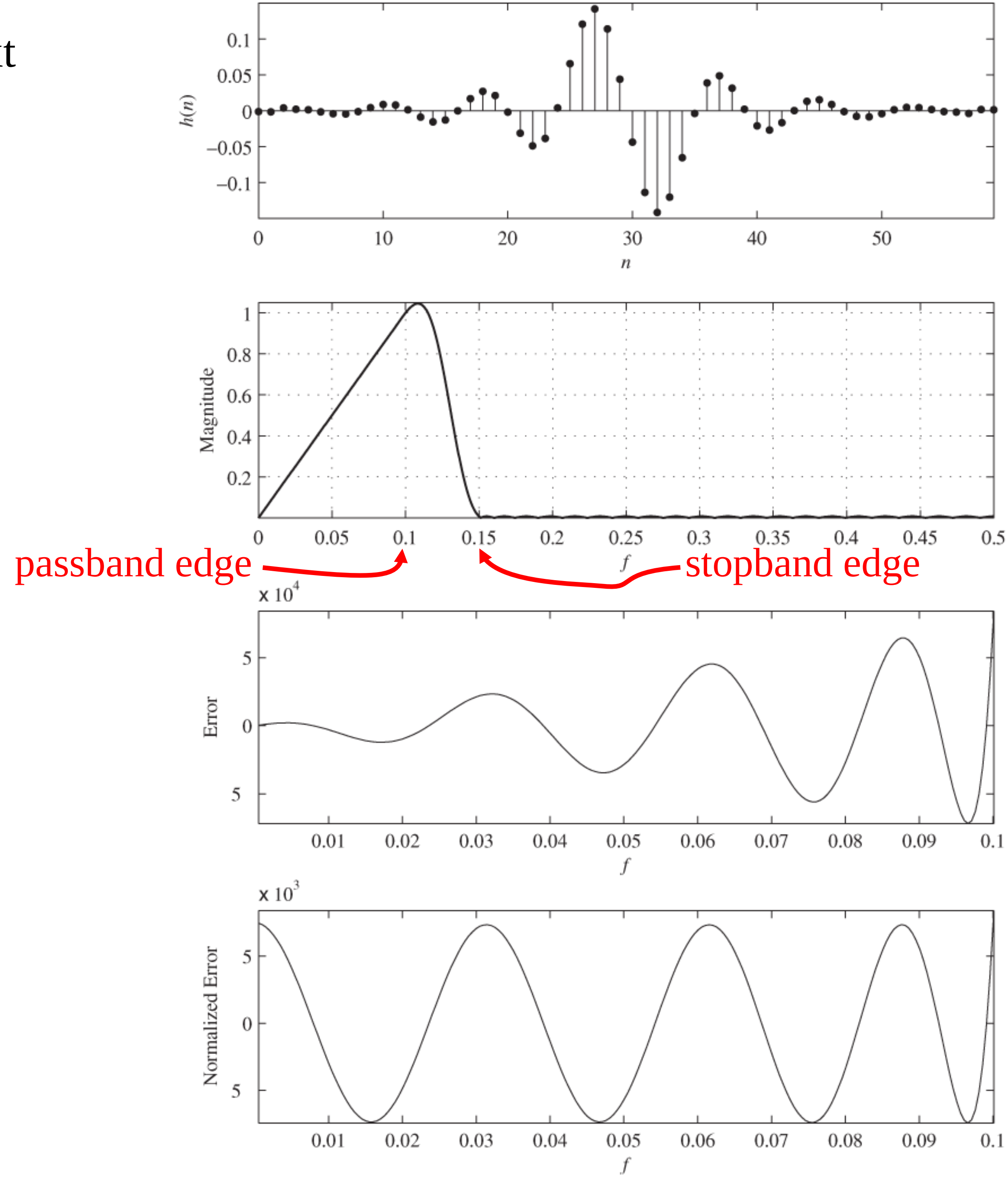
b = firpm(n,f,a,w,'ftype') specify a filter type, where 'ftype' is

- **'differentiator'**, for type III and type IV filters, using a special weighting technique

  For nonzero amplitude bands, it weights the error by a factor of 1/f so that the error at low frequencies is much smaller than at high frequencies. For FIR differentiators, which have an amplitude characteristic proportional to frequency, these filters minimize the maximum relative error (the maximum of the ratio of the error to the desired amplitude).

Example 10.2.5 in Text

$M = 60$



# Design of Equiripple FIR Hilbert Transformers

As we saw in the section on Equivalent Lowpass Signals… a Hilbert Transform is often used in generating the result.

So… trying to mimic this with a DT filter we have

$$H_d^{\mathrm{f}}(\omega) = \begin{cases} -j, & 0 < \omega \le \pi \\ j, & -\pi < \omega < 0 \end{cases} \qquad (2\pi\text{-periodic})$$

**Desired**

To explore this… compute its impulse response

$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d^{\mathrm{f}}(\omega) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} je^{j\omega n} d\omega - \int_{0}^{\pi} je^{j\omega n} d\omega\right]$$

$$= \begin{cases} \dfrac{2}{\pi}\dfrac{\sin^2(\pi n/2)}{n}, & n \neq 0 \\ 0, & n = 0 \end{cases}$$

**Antisymmetric: $h_d[-n] = -h_d[n]$**

Both antisymmetric Cases have $H(0) = 0$ and $M$ Odd gives $H(\pi) = 0$

… which are troublesome for the all-pass nature of the HT.

So… we usually spec lower and upper cutoff frequencies for the desired *amplitude* response:

$$H_{dr}^{f}(\omega) = 1, \quad 0 < \omega_l \leq |\omega| \leq \omega_u < \pi$$

## Design of Equiripple Differentiators via firpm

b = firpm(n,f,a,w,'ftype') specify a filter type, where 'ftype' is

- **'hilbert'**, for linear-phase filters with odd symmetry (type III and type IV) The output coefficients in b obey the relation b(k) = –b(n+2 –k), k = 1, ...,n+1. This class of filters includes the Hilbert transformer, which has a desired amplitude of 1 across the entire band.

  For example,

  h = firpm(30,[0.1 0.9],[1 1],'hilbert');

  designs an approximate FIR Hilbert transformer of length 31.

# Example 10.2.6 in Textbook

$M = 31$

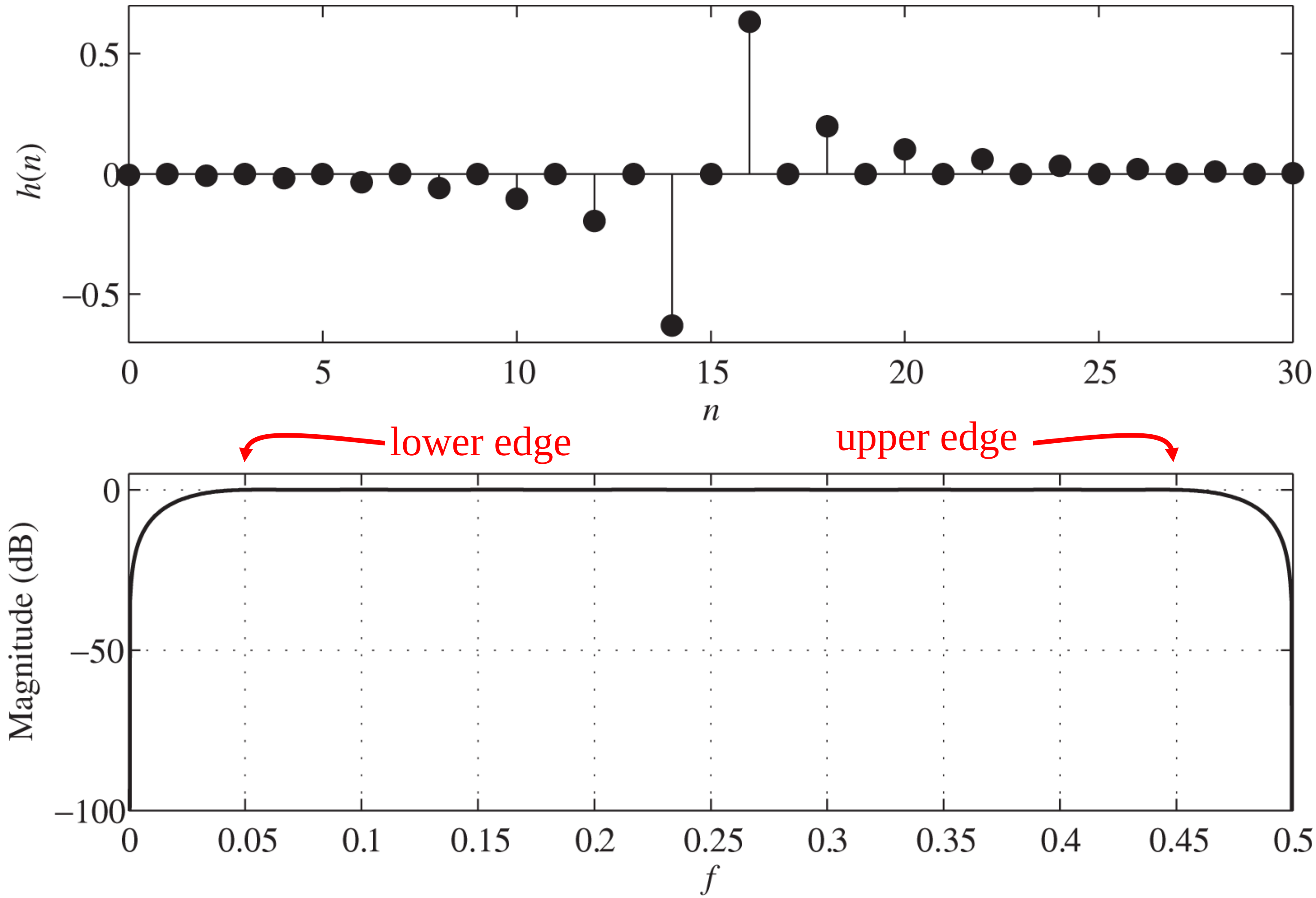



**Figure 10.2.24** Frequency of FIR Hilbert transform filter in Example 10.2.6.