EEO 401
Digital Signal Processing
Prof. Mark Fowler

Note Set #3

• Convolution & Impulse Response – Review
• Reading Assignment: Sect. 2.3 of Proakis & Manolakis
**Convolution for LTI D-T systems**

We are trying to find $y(t)$… when all ICs = 0

i.e. no stored “energy”

Before we can find the output… we need something first:

**Impulse Response**

The impulse response $h[n]$ is what “comes out” when $\delta[n]$ “goes in” w/ ICs=0

Note: If system is causal, then $h[n] = 0$ for $n < 0$
The impulse response $h[n]$ uniquely describes the system... so we can identify the system by specifying its impulse response $h[n]$.

Thus, we often show the system using a block diagram with the system’s impulse response $h[n]$ inside the box representing the system:

![Block Diagram](image)

Because impulse response $h[n]$ is only defined for LTI systems, if you see a box with the symbol $h[n]$ inside it you can assume that the system is an LTI system.

![Block Diagram](image)
Q: How do we use $h[n]$ to find the Zero-State Response?

A: “Convolution”  We’ll go through three analysis steps that will derive “The General Answer” that convolution is what we need to do to find the zero-state response

After that… we won’t need to re-do these steps… we’ll just “Do Convolution”

**Step 1:** Using **time-invariance** we know: $\delta[n-i] \rightarrow h[n]$ (w/ ICs = 0) $\rightarrow h[n-i]$ Shifted input gives shifted output

**Step 2:** Use “homogeneity” part of linearity: $x[i] \delta[n-i] \rightarrow h[n]$ (w/ ICs = 0) $\rightarrow x[i]h[n-i]$ The input is a function of $n$ so we view $x[i]$ as a fixed number for a given $i$

So… we scale the output by the same fixed number
Let’s see step 2… for a specific input:

\[ x[i] \delta[n-i] \xrightarrow{h[n]} x[i]h[n-i] \]

(w/ ICs = 0)
**Step 3: Use “additivity” part of linearity**

In Step 2 we looked at inputs like this:

\[ x[i] \delta[n-i] \]

\[ \rightarrow \]

For each \( i \), a different input \( \Rightarrow \) For each \( i \), a different response

**Now we use the additivity part of linearity:**

Put the **Sum of Those Inputs** In \( \Rightarrow \) Get the **Sum of Their Responses** Out

\[ \sum_{i=-\infty}^{\infty} x[i] \delta[n-i] \rightarrow \sum_{i=-\infty}^{\infty} x[i] h[n-i] \]

But… what is this??
On the next slide we show that it is the desired input signal \( x[n] \)!
Let’s see step 3 for a specific input:

\[ \sum_{i=-\infty}^{\infty} x[i] \delta[n - i] \]

Note: The Sum of these “x-weighted” impulses gives \( x[n]!! \)
So... what we’ve seen is this:

Input: \[ \sum_{i=-\infty}^{\infty} x[i] \delta[n-i] \]

Output: \[ \sum_{i=-\infty}^{\infty} x[i] h[n-i] \]

= \[ x[n] \]

Or in other words... we’ve derived an expression that tells what comes out of a D-T LTI system with input \( x[n] \):

\[ y[n] = \sum_{i=-\infty}^{\infty} x[i] h[n-i] \]

\[ y[n] = x[n] * h[n] \]

So... now that we have derived this result we don’t have to do these three steps... we “just” use this equation to find the zero-state output:

\[ y[n] = \sum_{i=-\infty}^{\infty} x[i] h[n-i] \]

Note: In your Signals & Systems course you should have learned how to *do* convolution.... You should review that!
Big Picture

For a LTI D-T system in zero state characterized by impulse response $h[n]$, we can analytically find the output when the input is $x[n]$ by performing the convolution between $x[n]$ and $h[n]$.

\[ y[n] = \sum_{i=-\infty}^{\infty} x[i]h[n-i] = \sum_{i=-\infty}^{\infty} h[i]x[n-i] \]

What if the LTI system is causal? \( h[n] = 0 \ \forall n < 0 \)

\[ y[n] = \sum_{i=-\infty}^{n} x[i]h[n-i] = \sum_{i=0}^{\infty} h[i]x[n-i] \]

What if the input “starts” @ $n = 0$? \( x[n] = 0 \ \forall n < 0 \)

\[ y[n] = \sum_{i=0}^{\infty} x[i]h[n-i] = \sum_{i=-\infty}^{n} h[i]x[n-i] \]

What if the LTI system is causal and input “starts” @ $n = 0$?

\[ y[n] = \sum_{i=0}^{n} x[i]h[n-i] = \sum_{i=0}^{n} h[i]x[n-i] \]
Convolution Properties

These are things you can exploit to make it easier to solve problems

1. **Commutativity** \( x[n] * h[n] = h[n] * x[n] \)
   \( \Rightarrow \) You can choose which signal to “flip”

2. **Associativity** \( x[n] * (v[n] * w[n]) = (x[n] * v[n]) * w[n] \)
   \( \Rightarrow \) Can change order → sometimes one order is easier than another

3. **Distributivity** \( x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n] \)
   \( \Rightarrow \) may be easier to split complicated system \( h[n] \) into sum of simple ones

OR.. \( \Rightarrow \) we can split complicated input into sum of simple ones
   (nothing more than “linearity”)

4. **Convolution with impulses** \( x[n] * \delta[n - q] = x[n - q] \)

This one is **VERY** easy to see using the graphical convolution steps.

**TRY IT!!**
Checking for Stability via the Impulse Response

An LTI system is BIBO stable if and only if its impulse response is “absolutely summable”:

\[ \sum_{n=-\infty}^{\infty} |h[n]| < \infty \]

Systems w/ Infinite-Duration & Finite-Duration Impulse Resp.

For simplicity of notation we focus on causal systems here.

\[ y[n] = \sum_{i=0}^{\infty} h[i]x[n-i] \]

In general, the impulse response has infinite duration

A system for which \( h[n] \) has infinitely many non-zero values is said to be an “infinite-duration impulse response (IIR) system.

A system for which \( h[n] \) has finitely many non-zero values is said to be an “finite-duration impulse response (FIR) system. Suppose \( h[n] = 0 \) for \( n < 0 \) and for \( n \geq M \) then the convolution sum becomes

\[ y[n] = \sum_{i=0}^{M-1} h[i]x[n-i] \]

This is said to be an order \( M \) FIR system