BINGHAMTON UNIVERSITY
State University of New York

EEO 401
Digital Signal Processing
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Note Set #28

- Linear Phase and FIR Filter Symmetry
- Reading: Sect. 10.2.1 of Proakis & Manolakis

Definition of Linear Phase

Usually we define the phase of $H^{\mathrm{f}}(\omega)$ based on writing it in *magnitude* and phase form like this:

$$H^{\mathrm{f}}(\omega) = \left|H^{\mathrm{f}}(\omega)\right| e^{j \arctan\left\{H^{\mathrm{f}}(\omega)\right\}}$$

Strictly ≥ 0

However, this causes some problems of discontinuous phase when $H^{\mathrm{f}}(\omega)$ goes "through" zero on the complex plane.

This causes some problems but we can fix them if we are careful!

Example: (from Porat's DSP Book) $H^z(z) = 1 - z^{-1}$

$$H^{\mathrm{f}}(\omega) = 1 - e^{-j\omega} = \left(e^{j\omega/2} - e^{-j\omega/2}\right)e^{-j\omega/2}$$

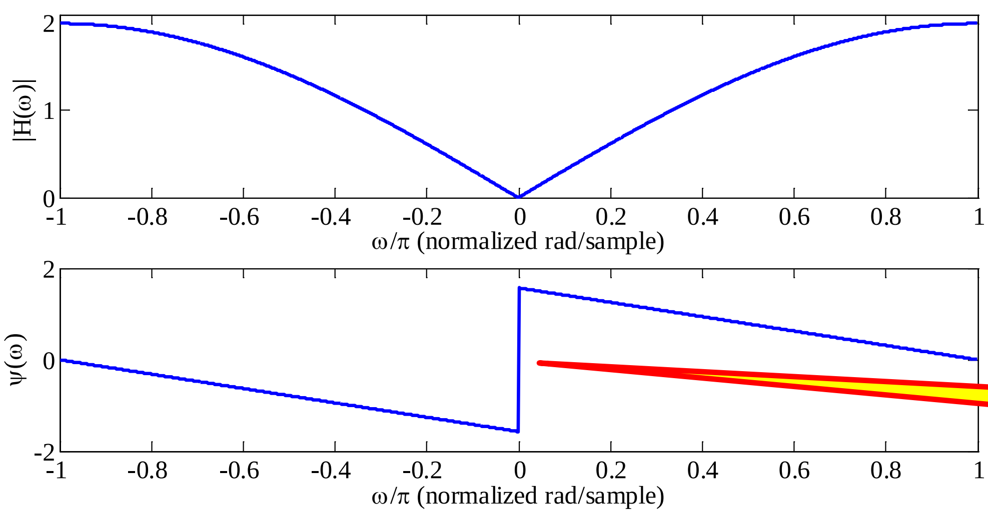
$$= 2j\sin(\omega/2)e^{-j\omega/2} = 2\sin(\omega/2)e^{j(\pi/2-\omega/2)}$$

< 0 for –π < ω < 0

$$\left|H^{\mathrm{f}}(\omega)\right| = \left|2\sin(\omega/2)\right|$$

Causes jump of π

$$\psi(\omega) = \arctan\{H^{\mathrm{f}}(\omega)\} = \begin{cases} -\dfrac{\pi}{2} - \dfrac{\omega}{2}, & -\pi < \omega < 0 \\ \dfrac{\pi}{2} - \dfrac{\omega}{2}, & -\pi < \omega < 0 \end{cases}$$



Computed using freqz and plotted using abs & angle commands

Not really a vertical there

This issue can be addressed by modifying our "magnitude" – phase view to be "amplitude" – phase view

Whereas "magnitude" is strictly non-negative… "amplitude" can be negative.

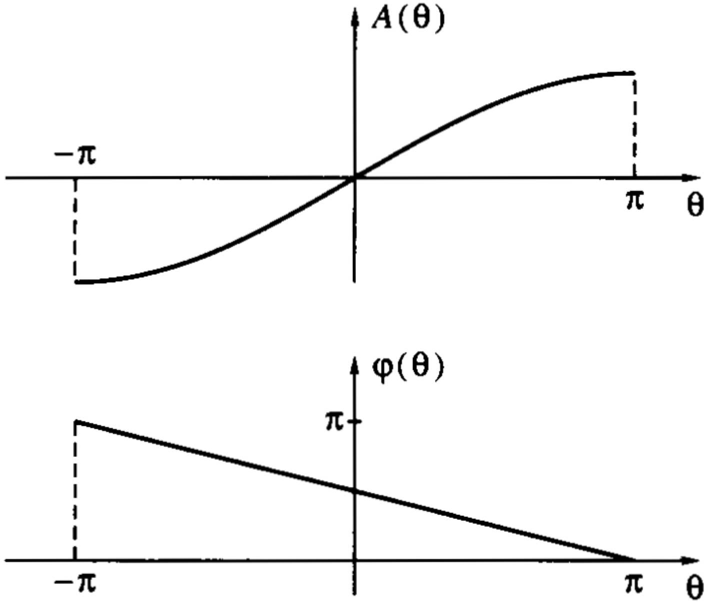
For our example: $H^z(z) = 1 - z^{-1}$

$$H^f(\omega) = 1 - e^{-j\omega} = \left(e^{j\omega/2} - e^{-j\omega/2}\right)e^{-j\omega/2}$$

$$= 2j\sin(\omega/2)e^{-j\omega/2} = 2\sin(\omega/2)e^{j(\pi/2-\omega/2)}$$

$A(\omega)$ $\phi(\omega)$

Fig. from Porat's DSP Book



Thus we can fix these jumps of π by using this amplitude rather than magnitude

- **Usually, though we just plot the regularly computed phase and visually account for these jumps-by-π**

Even with this new definition there can still be jumps by 2π

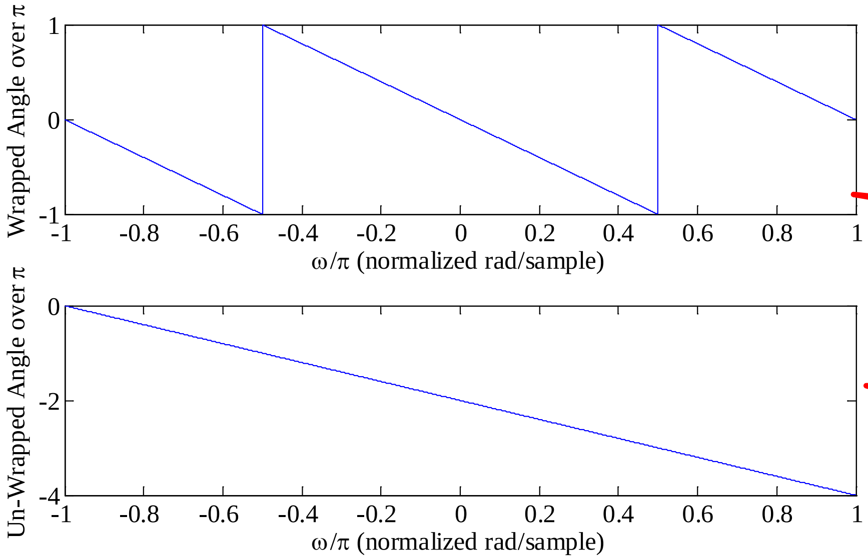
- These can be thought of as occurring when the phase function crosses from –π to +π (or vice versa).
- Keep in mind that any angle remains effectively the same with an integer multiple of 2π added or subtracted
- Thus, these jumps can be fixed "by eye" or in MATLAB via the unwrap command

Simplistic Example of using "unwrap" command

$$H^z(z) = z^{-2} \quad \Rightarrow \quad H^f(\omega) = e^{-j2\omega} \quad \Rightarrow \quad \angle H^f(\omega) = -2\omega$$

As ω goes past π/2 the angle goes more negative than –π

Numerical computations "wrap" these back into –π to + π



Computed using "angle(H)"

Computed using "unwrap(angle(H))"

So finally we get to the point where we can define what "linear phase" means!

After correcting for the π and 2π jumps and also adding/subtracting 2π to put the angle at $\omega = 0$ in $[0, \pi)$….

We say the filter is linear phase if the resulting phase is $-\omega\tau_p$ such that we can write

$$H^{f}(\omega) = A(\omega)e^{-j\omega\tau_p}$$

for $A(\omega)$ a real amplitude function and τ_p a real number

If… τ_p is an integer then this linear phase will impart a delay of τ_p samples
τ_p is not an integer then this linear phase will impart an "interpolated" delay of τ_p samples

Symmetry and Linear Phase for FIR Filters

An FIR filter of length M for input $x[n]$ and output $y[n]$ is given by

$$y[n] = b_0 x[n] + b_1 x[n-1] + \cdots + b_{M-1} x[n-M+1]$$
$$= \sum_{k=0}^{M-1} b_k x[n-k]$$

Filter Coefficients

Comparing to the convolution form

$$y[n] = \sum_{k=0}^{M-1} h[k] x[n-k]$$

we see that the filter coefficients are the non-zero values of the impulse response

$$h[n] = \begin{cases} b_n, & n = 0,1,\ldots,M-1 \\ 0, & otherwise \end{cases}$$

$$H^z(z) = \sum_{k=0}^{M-1} h[k] z^{-k} = \sum_{k=0}^{M-1} b_k z^{-k}$$

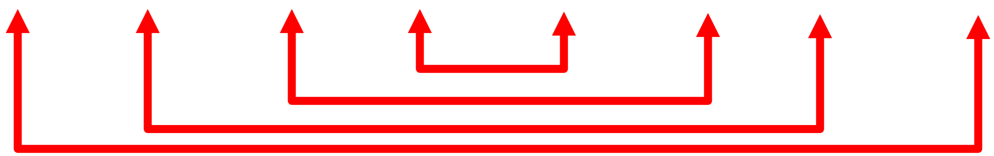
$$H^f(\omega) = \sum_{k=0}^{M-1} h[k] e^{-j\omega k} = \sum_{k=0}^{M-1} b_k e^{-j\omega k}$$

An FIR filter has linear phase if its impulse response (filter coefficients) has symmetry:

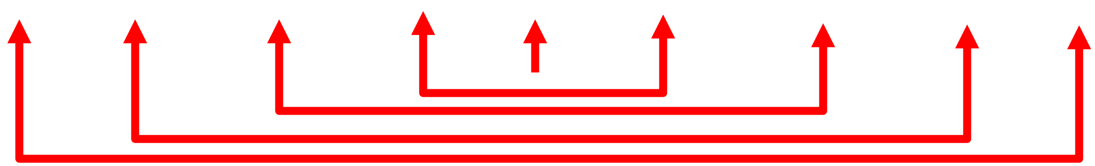
$$h[n] = \pm h[M-1-n], \quad n = 0, 1, \cdots, M-1$$

+: Symmetric
–: Anti-Symmetric

$h[0] \quad h[1] \quad h[2] \quad h[3] \quad h[4] \quad h[5] \quad h[6] \quad h[7] \quad M = 8$ **Even Length**



$h[0] \quad h[1] \quad h[2] \quad h[3] \quad h[4] \quad h[5] \quad h[6] \quad h[7] \quad h[8] \quad M = 9$ **Odd Length**



For Odd Length, Anti-Symmetric need *h*[(M-1)/2] = 0

Look at this simple example: $h[n] = h[M-1-n], \quad n = 0,1,\cdots,M-1$

$h[0]$	$h[1]$	$h[2]$	$h[3]$	$h[4]$	$h[5]$	$h[6]$	$h[7]$
1	2	3	4	4	3	2	1

$M = 8$ **Even Length**

$$H^z(z) = \sum_{k=0}^{M-1} h[k] z^{-k}$$

$$= 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 4z^{-4} + 3z^{-5} + 2z^{-6} + 1z^{-7}$$

$$= z^{-7/2}[1z^{7/2} + 2z^{7/2}z^{-1} + 3z^{7/2}z^{-2} + 4z^{7/2}z^{-3} + 4z^{7/2}z^{-4} \ldots$$

$$\cdots + 3z^{7/2}z^{-5} + 2z^{7/2}z^{-6} + 1z^{7/2}z^{-7}]$$

$$= z^{-7/2}[1(z^{7/2} + z^{-7/2}) + 2(z^{5/2} + z^{-5/2}) \ldots$$

$$\cdots + 3(z^{3/2} + z^{-3/2}) + 4(z^{1/2} + z^{-1/2})]$$

Converting to frequency response, the z terms in parentheses get converted into cosines via Euler… so the part in the [ ] becomes a real-valued amplitude function and we get

$$H^f(\omega) = H_r(\omega) e^{-j\omega(M-1)/2}$$

Linear Phase

We can now do a similar thing in general for the symmetric/anti-symmetric FIR:

Using the symmetry/anti-symmetry condition in the transfer function gives

Odd Length *M*

$$H^z(z) = z^{-(M-1)/2}\left[h\left[\frac{M-1}{2}\right] + \sum_{k=0}^{(M-3)/2} h[k]\left(z^{(M-1-2k)/2} \pm z^{-(M-1-2k)/2}\right)\right]$$

Even Length *M*

$$H^z(z) = z^{-(M-1)/2}\left[\sum_{k=0}^{(M/2-1)} h[k]\left(z^{(M-1-2k)/2} \pm z^{-(M-1-2k)/2}\right)\right]$$

Following steps similar to our example we get

$$h[n] = h[M-1-n] \Rightarrow H^f(\omega) = H_r(\omega)e^{-j\omega(M-1)/2}$$

$$h[n] = -h[M-1-n] \Rightarrow H^f(\omega) = H_r(\omega)e^{-j\omega(M-1)/2+\pi/2}$$

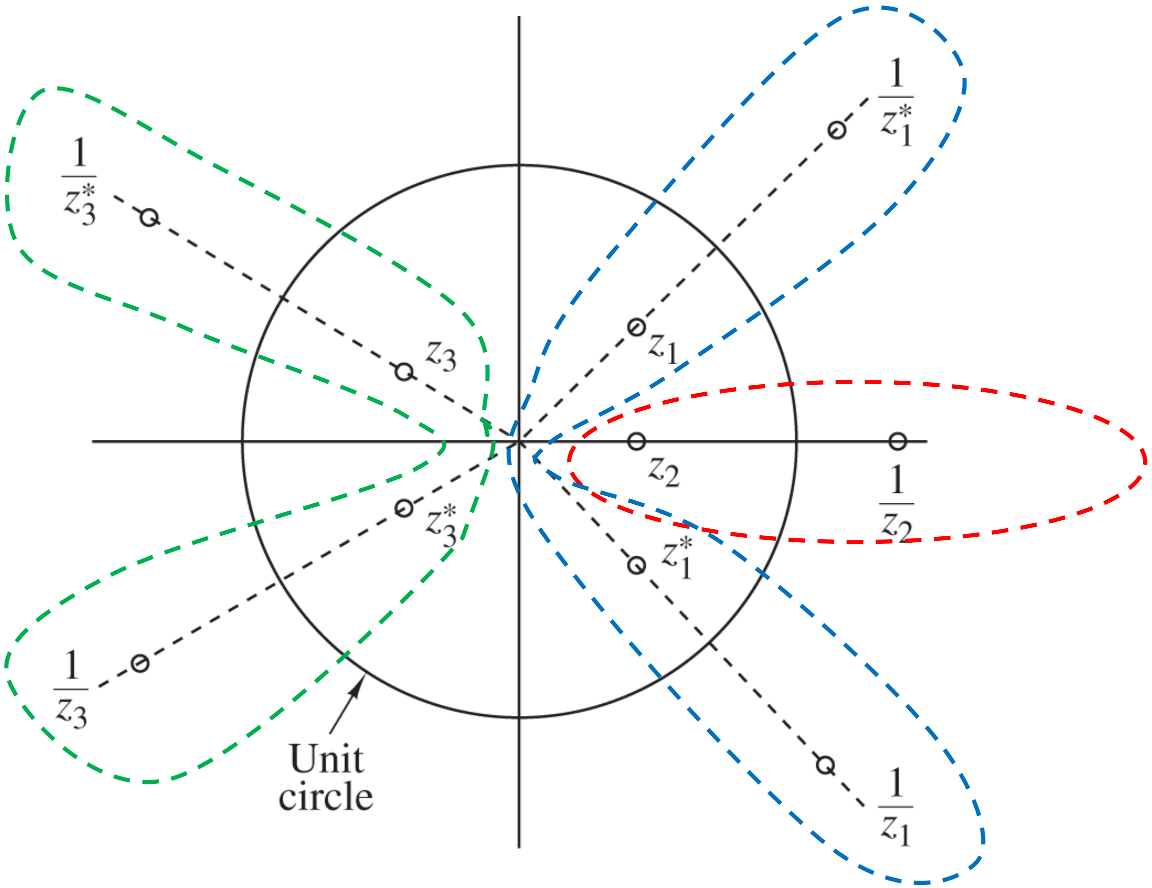
So… both types of symmetry give linear phase!

See textbook for form for $H_r(\omega)$

Symmetry of zero locations for linear-phase FIR filters

Start with the transfer function form for the symmetric FIR filters…
Sub in z^{-1} for z & multiply both sides by $z^{-(M-1)}$ gives

$$z^{-(M-1)}H(z^{-1}) = \pm H(z)$$



Uses of Symmetric vs Anti-Symmetric FIR Filters

Note that $$H^{\mathrm{f}}(0) = \sum_{n=0}^{M-1} h[n]$$

Thus… if the filter is Anti-Symmetric $h[n] = -h[M-1-n]$

we must have $H^{\mathrm{f}}(0) = 0$

So… Anti-Symmetric FIR filters are not suited for LP filters.

Symmetric FIR filters are suited for LP filters as well as HP, etc.