BINGHAMTON UNIVERSITY

State University of New York

EEO 401
Digital Signal Processing
Prof. Mark Fowler

<u>Note Set #23</u>

- Windows for Spectral Analysis of Signals
- Reading Assignment: Sect. 7.4 of Proakis & Manolakis
  Ch. 6 of Porat's Book

Common Windows

Porat Section 6.3

Desirable Window Properties

We've seen that to minimize the impact of a window we need the DTFT of the window $W^f(\theta)$ to have:

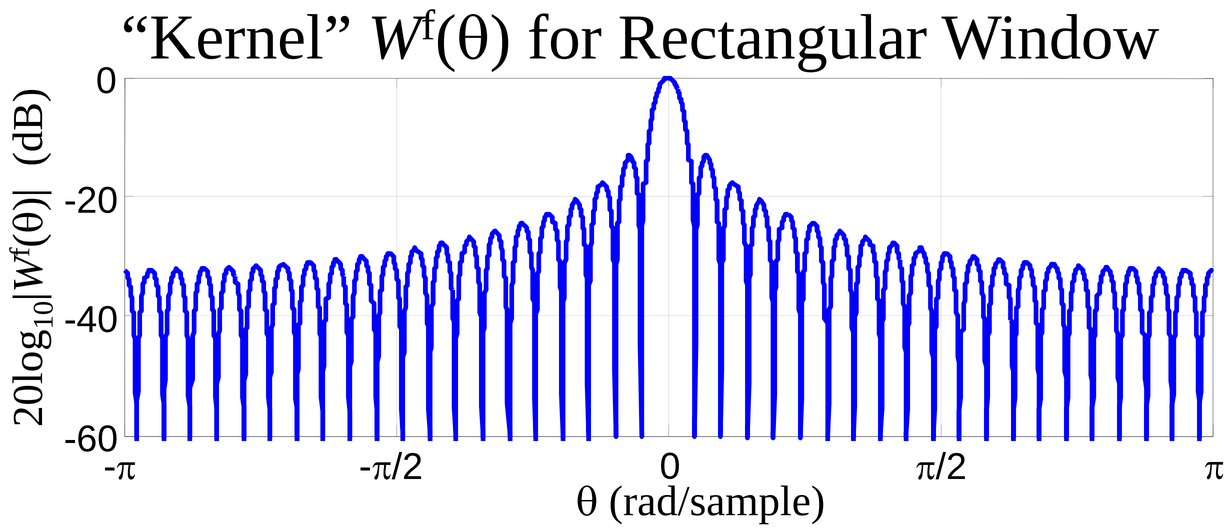
- Narrow Mainlobe
  - Mainlobe Width *usually* measured "zero-to-zero"
- Small Sidelobe Levels
  - Measured in dB relative to mainlobe peak
  - Care about "Highest Sidelobe" & "Drop-off Rate"

We'll see that there is an inherent trade-off between these two desires:

Lowering the Sidelobes Causes a Widening of the Mainlobe

Rectangular Window

This is what you get if you don't explicitly apply some other type of window – it is due to the fact that you have only *N* signal samples available.



- Mainlobe Width = 4π/N

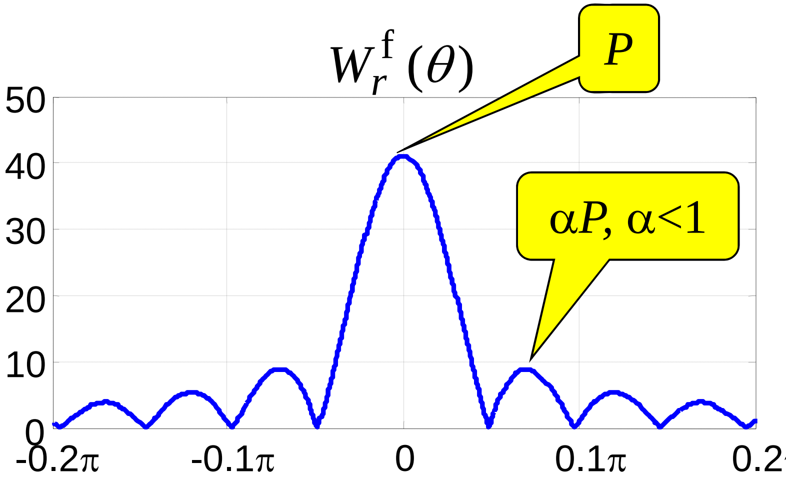
Good!!!

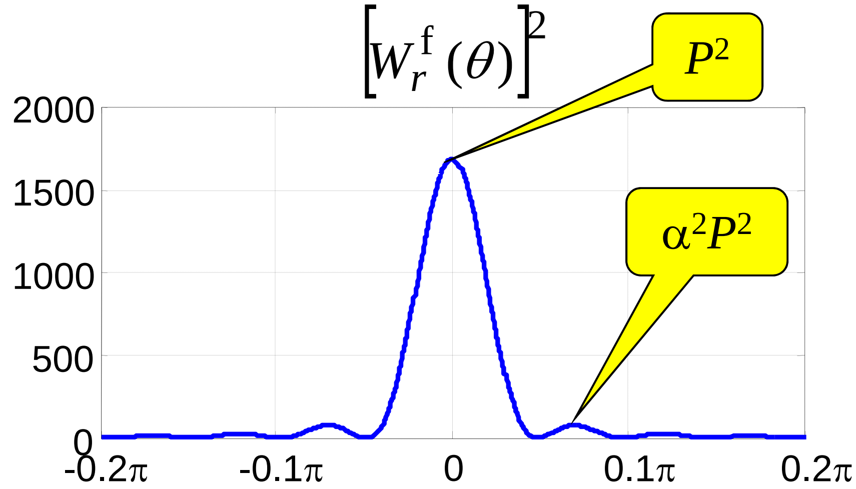
- Sidelobe Levels
  - Largest Sidelobe = –13 dB
  - Sidelobe Drop-off Rate = –6 dB/octave (except near ± π)

Bad!!!
Need to get these lower!!!
But HOW????

Bartlett Window

Inspiration: Square the (non-dB) rect. kernel

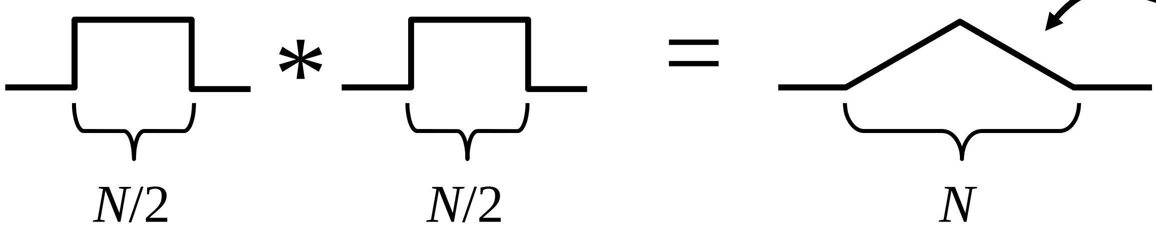




Sidelobe Ratio = $\alpha P/P = \alpha$

Sidelobe Ratio = $\alpha^2 P^2/P^2 = \alpha^2 < \alpha$

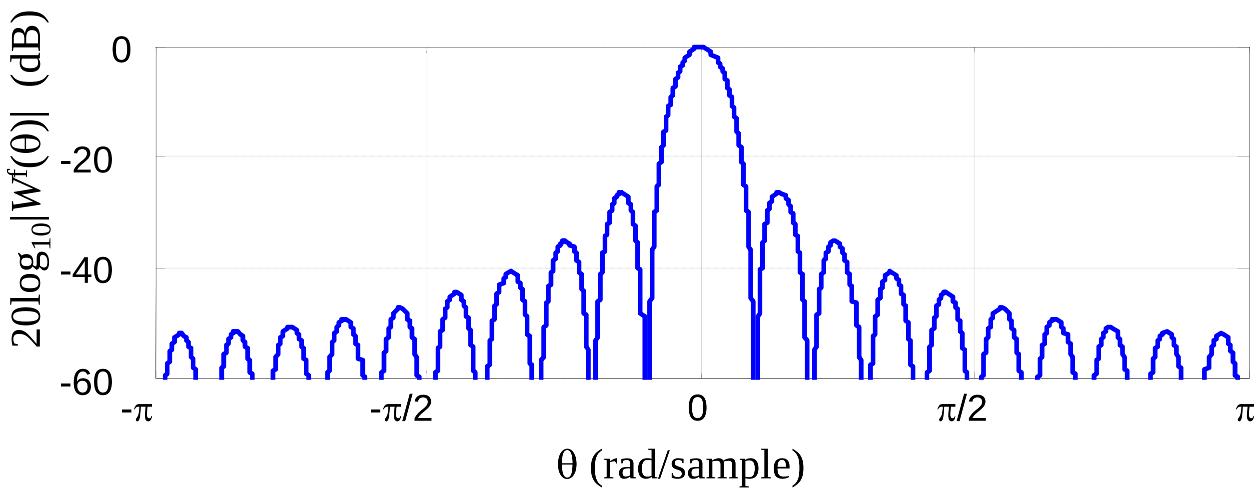
So… in time-domain this corresponds to convolving 2 rect. windows:



Bartlett Window
Also called
Triangular
Window

Bartlett Window (pt. 2)

“Kernel” $W^f(\theta)$ for Bartlett Window



- Mainlobe Width = $8\pi/(N+1)$

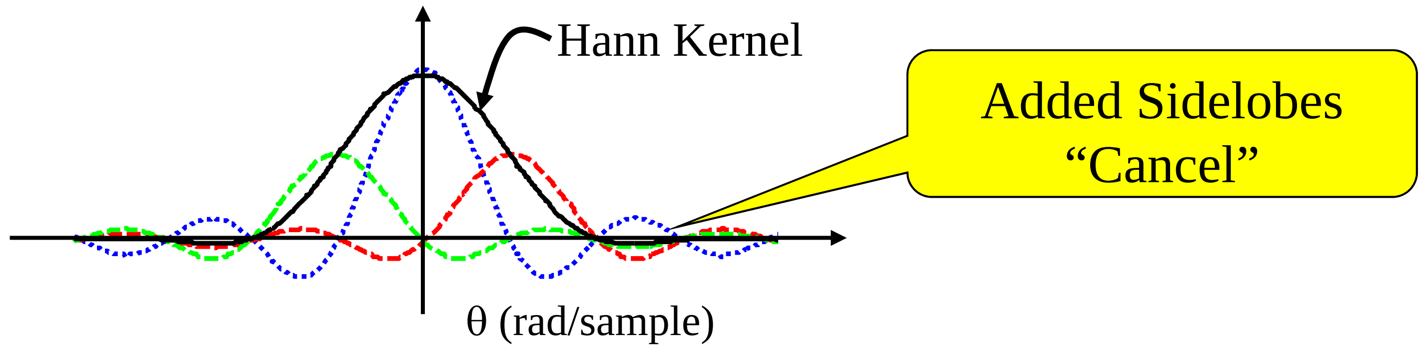
≈ 2 ×Wider Than Rect

- Sidelobe Levels
  - Largest Sidelobe = –27 dB
  - Sidelobe Drop-off Rate = –12 dB/octave (except near ± π)

Better than Rect
-27 dB vs. –13 dB
-12 dB/oct vs. -6 dB/oct

Hann Window (also called Hanning)

Inspiration: "Add" three shifted (non-dB) rect. kernels together to try to cancel sidelobes:



$$W_{\mathrm{hn}}^{\mathrm{f}}(\theta) = \underbrace{0.5W_{\mathrm{r}}^{\mathrm{f}}(\theta)} - \underbrace{0.25W_{\mathrm{r}}^{\mathrm{f}}\left(\theta - \frac{2\pi}{N-1}\right)} - \underbrace{0.25W_{\mathrm{r}}^{\mathrm{f}}\left(\theta + \frac{2\pi}{N-1}\right)}$$

Use DTFT
Freq. Shift Property

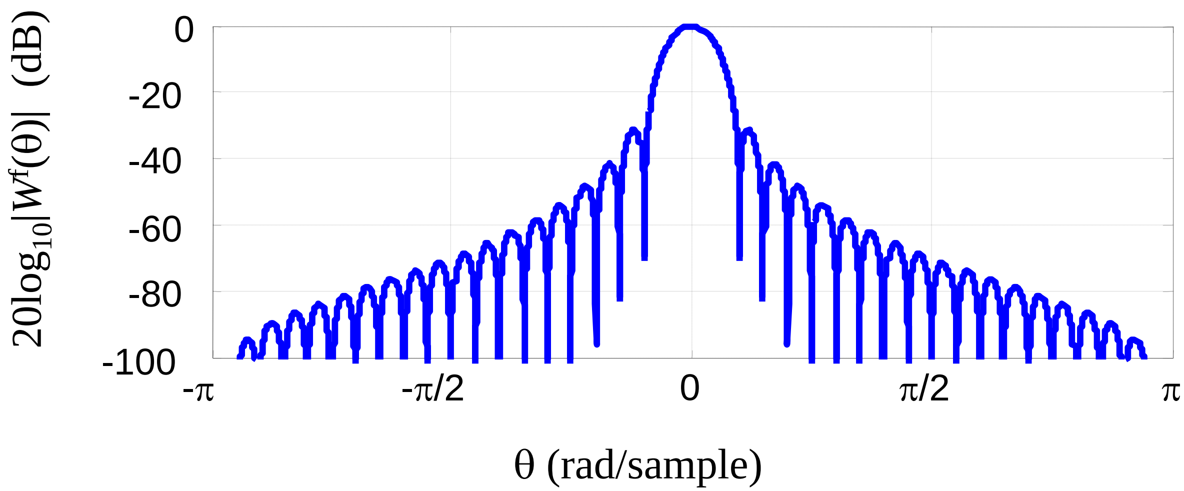
$$w_{\mathrm{hn}}[n] = 0.5 - \overbrace{0.25e^{j2\pi n/(N-1)}} - \overbrace{0.25e^{-j2\pi n/(N-1)}}, \quad 0 \le n \le N-1$$

$$= 0.5\left[1 - \cos\{2\pi n/(N-1)\}\right]$$

Hann Window

Hann Window (pt. 2)

"Kernel" $W^f(\theta)$ for Hann Window



- Mainlobe Width = $8\pi/(N)$

2 ×Wider Than Rect

- Sidelobe Levels
  - – Largest Sidelobe = –32 dB
  - – Sidelobe Drop-off Rate = –18 dB/octave (except near ± π)

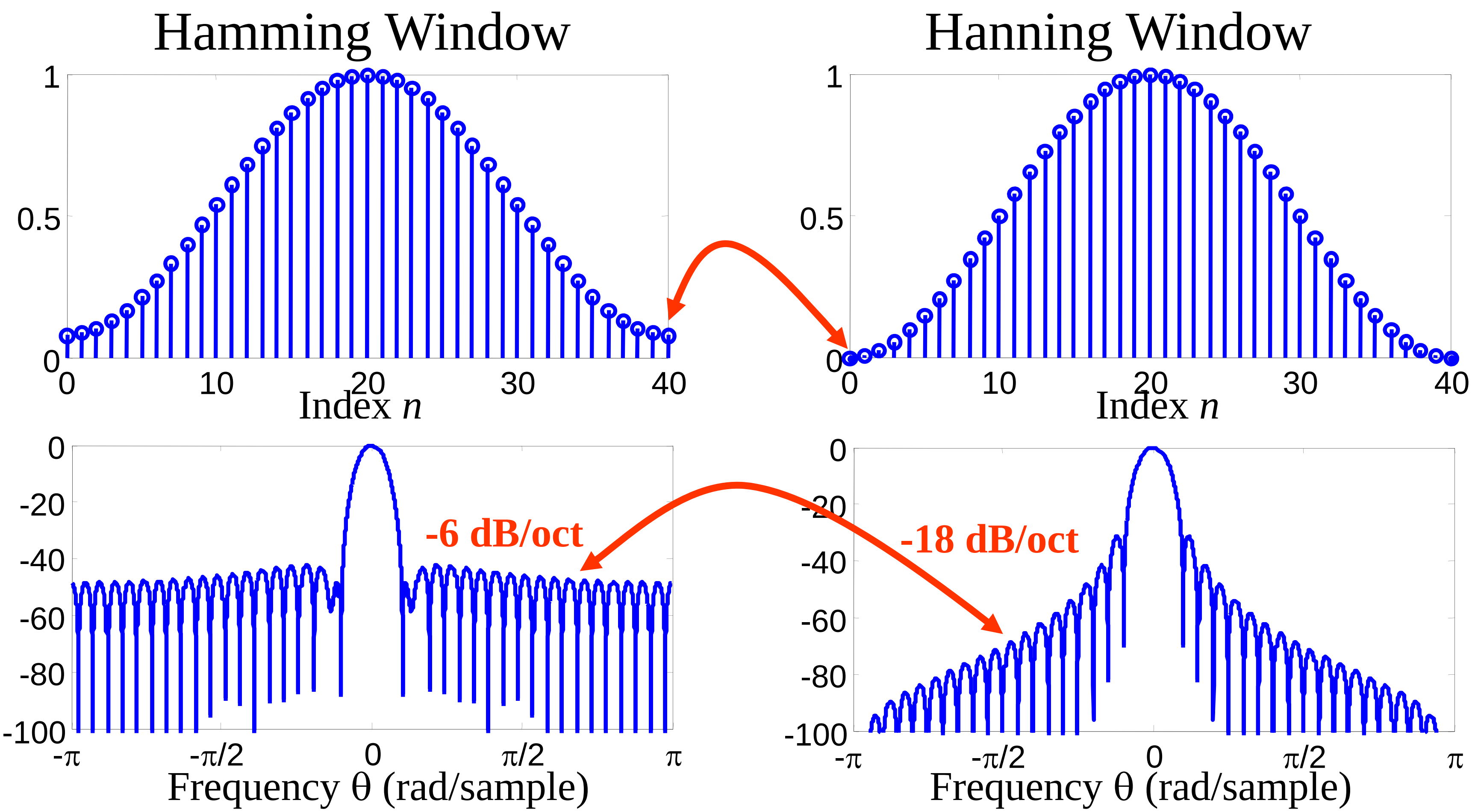
Even Better than Bartlett
-32 dB vs. –27 dB
-18 dB/oct vs. -12 dB/oct

Hamming Window

Inspiration: Tweak Hann coefficients to get lower "highest SL"

$$W_{\mathrm{hm}}^{\mathrm{f}}(\theta) = 0.54W_{\mathrm{r}}^{\mathrm{f}}(\theta) - 0.23W_{\mathrm{r}}^{\mathrm{f}}(\theta - \frac{2\pi}{N-1}) - 0.23W_{\mathrm{r}}^{\mathrm{f}}(\theta + \frac{2\pi}{N-1})$$

$$\boxed{w_{\mathrm{hn}}[n] = 0.54 - 0.46\cos\{2\pi n/(N-1)\}}$$



Hamming Window (pt. 2)

2 ×Wider Than Rect
(same as Hanning)

- Mainlobe Width = $8\pi/(N)$

Even Better than Hanning
–43 dB vs. –32 dB

- Sidelobe Levels
  - Largest Sidelobe = –43 dB
  - Sidelobe Drop-off Rate = –6 dB/octave (except near $\pm\pi$)

As Bad as Rect!!
–6 dB vs. –6 dB

Note: Both Rect & Hamming have –6 dB/oct drop-off
Note also: Both are discontinuous at window edge in time-domain

Drop-Off Rate & Discontinuity Order

Definition: If the window's time-domain function is such that up to its **(p-1)th** derivative (but no higher) is continuous, then we say that the signal has ***p-order continuity***.

Ex. Rectangular Window has 0-order continuity
Triangular Window has 1-order continuity
Hamming Window has 0-order continuity

Result: A window that has *continuity of order p* will (generally) have a kernel that has a sidelobe drop-off rate of $-(p+1)$**6** dB/oct

Rectangular Window	has 0-order continuity:	- 6dB/oct
Hamming Window	has 0-order continuity:	- 6dB/oct
Triangular Window	has 1-order continuity:	- 12dB/oct
Hann Window	has 2-order continuity:	- 18dB/oct

Other Windows & Their Rationale

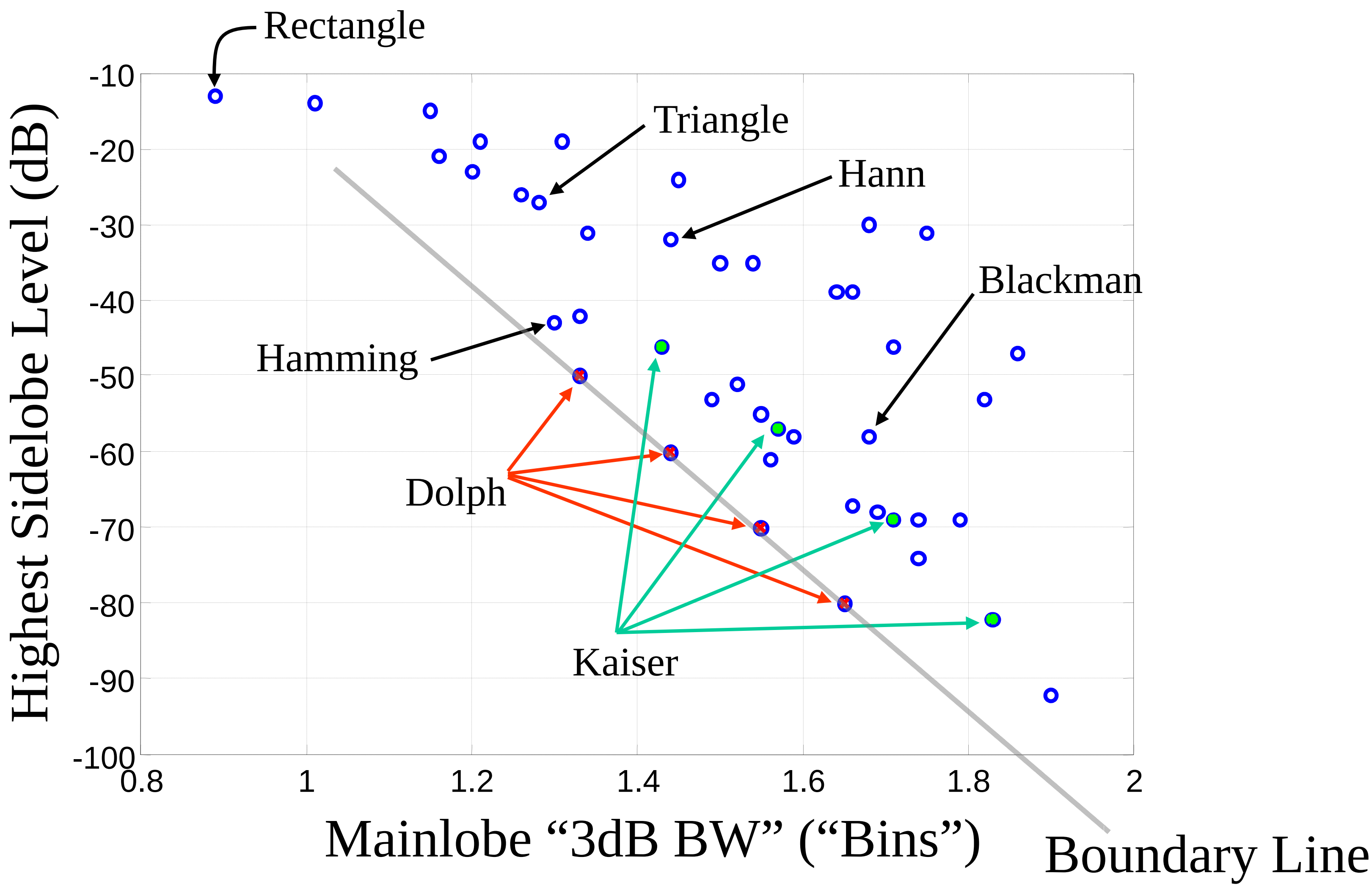
Lots of effort has been focused on designing good windows.
Here are a few, with their design rationale and their "specs"

Blackman: "More Tweaking of Hann Coefficients"
ML Width = $12\pi/N$ SL Level = -57 dB Drop-Off = -18 dB/oct

Kaiser: "Minimize width for SL *energy* not exceeding spec'd % of total"
ML Width = variable SL Level = variable Drop-Off = -6 dB/oct

Dolph: "Minimize width for SL *level* not exceeding spec'd level"
ML Width = variable SL Level = variable Drop-Off = 0 dB/oct

Comparison of Windows



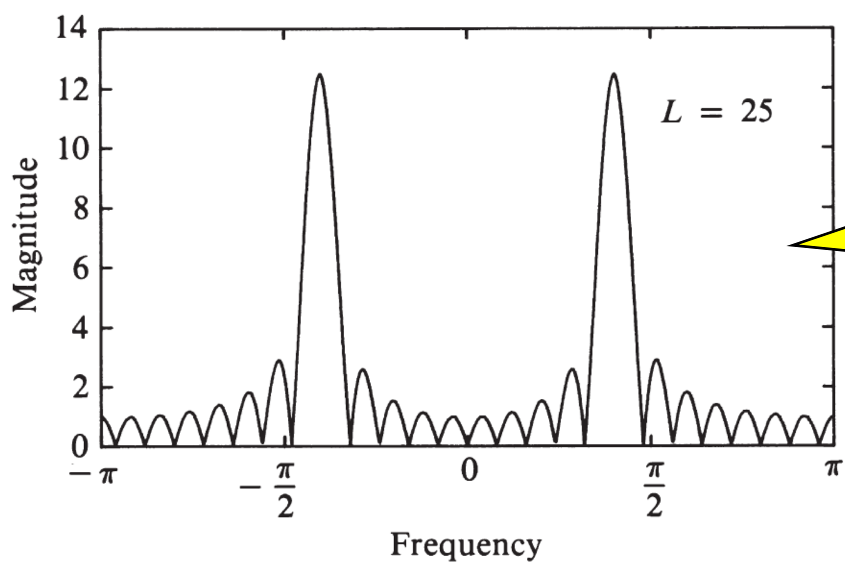
Data taken from table in F. J. Harris, “On the use of windows for harmonic analysis with the discrete Fourier transform,” *Proc. IEEE*, vol. 66, pp. 51 – 83, January 1978.

Some Examples from Proakis & Manolakis

Let L be the number of signal samples collected (i.e., "window size").
Let N be the number of points after zero-padding (i.e. "DFT size").

$$x[n] = \cos(\omega_o n), \quad n = 0,1,2,\ldots,L-1$$

Using Rectangular Window

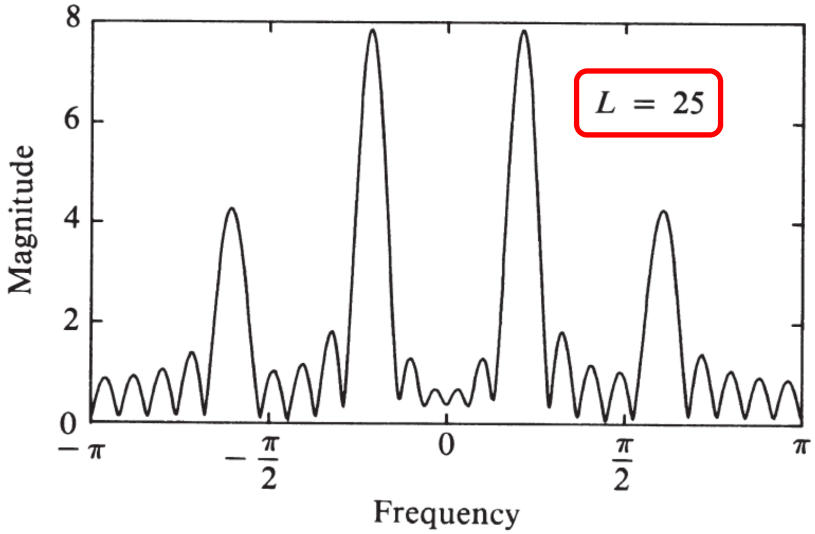


Plotted using "line" not "stem"… but looks smooth because N is sufficiently large!

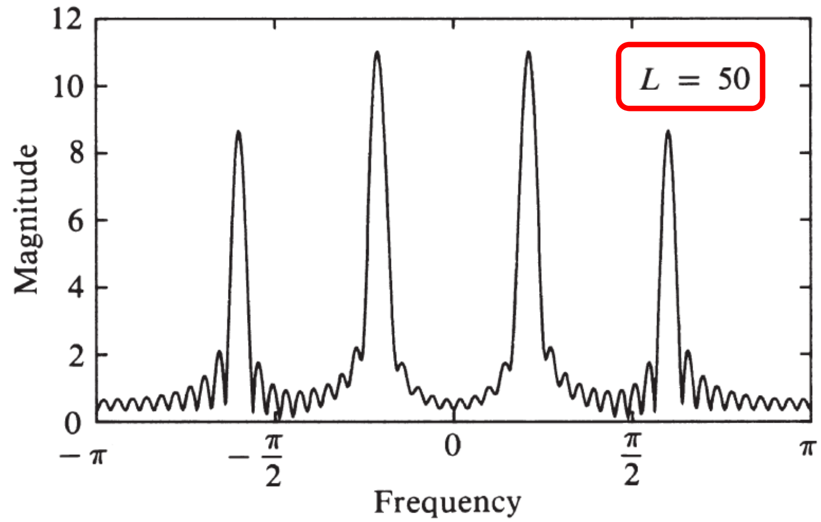
Figure 7.4.1 Magnitude spectrum for $L = 25$ and $N = 2048$, illustrating the occurrence of leakage.

$$x[n] = \cos(\omega_o n) + \cos(\omega_1 n) + \cos(\omega_2 n), \quad n = 0,1,2,\ldots,L-1$$

$\omega_o = 0.2\pi \quad \omega_o = 0.22\pi \quad \omega_o = 0.6\pi$ **N = 2048**

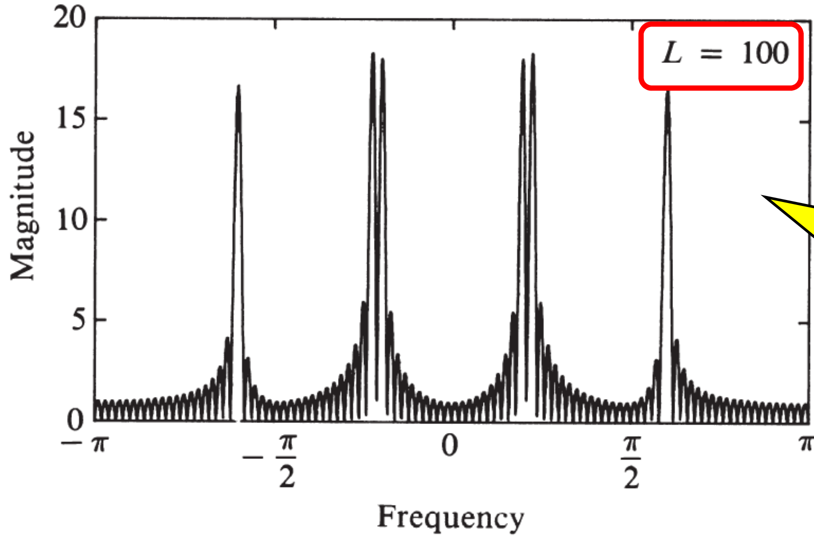


(a)



(b)

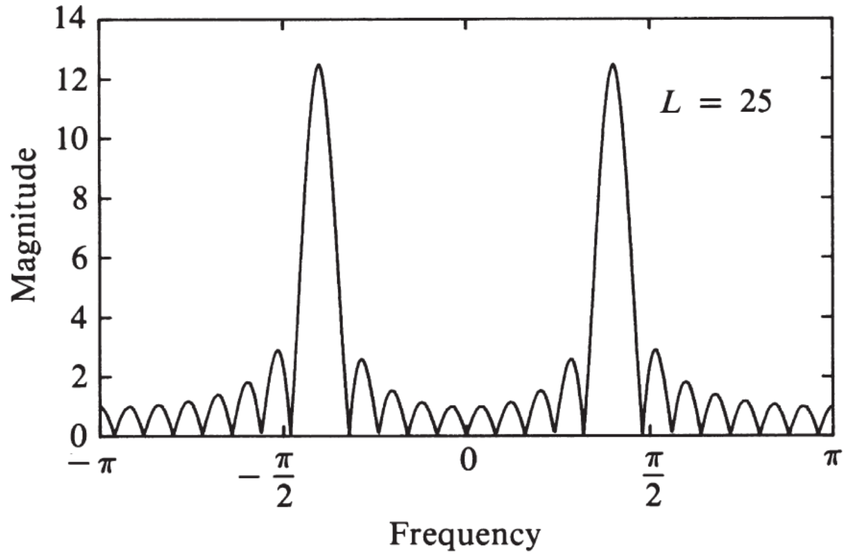
Using Rectangular Window



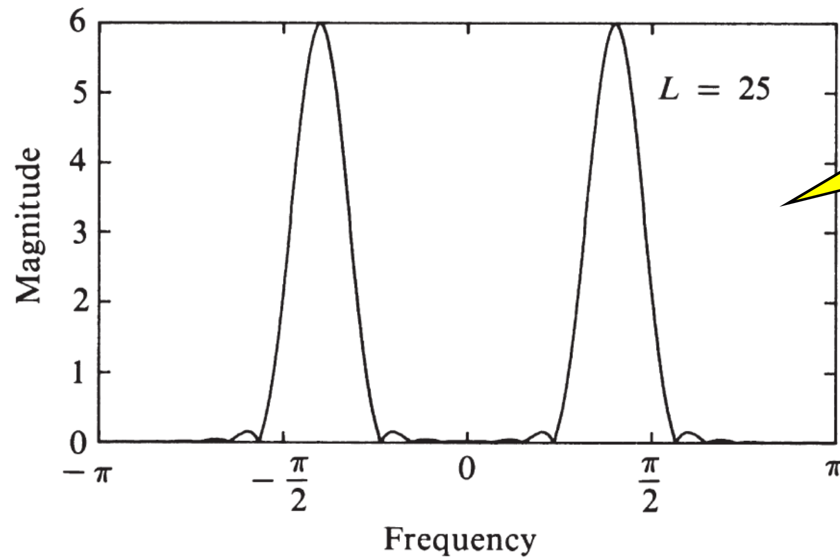
Increasing the amount of actual data allows one to see the separate spikes… because the window kernel gets narrower!

$$x[n] = \cos(\omega_o n), \quad n = 0,1,2,\ldots,L-1$$

Using Rectangular Window



Using Hanning Window



Hanning gives lower sidelobes at the expense of wider mainlobes!

$$x[n]=\cos(\omega_o n)+\cos(\omega_1 n)+\cos(\omega_2 n),\quad n=0,1,2,\ldots,L-1$$

$\omega_o = 0.2\pi \quad \omega_o = 0.22\pi \quad \omega_o = 0.6\pi$ **N = 2048**

Rectangular Window **Hanning Window**