BINGHAMTON
UNIVERSITY
State University of New York

EEO 401
Digital Signal Processing

Prof. Mark Fowler

Note Set #20

- Symmetries of the DFT
- Reading Assignment: Sect. 7.2 of Proakis & Manolakis

Circular Symmetries

In the ***context of DFT & IDFT*** we often need to discuss symmetries w.r.t. the circular nature of a signal.

Let this be an N-point signal (segment) : $x[n]$ — $= 0 \quad n < 0, n > N-1$

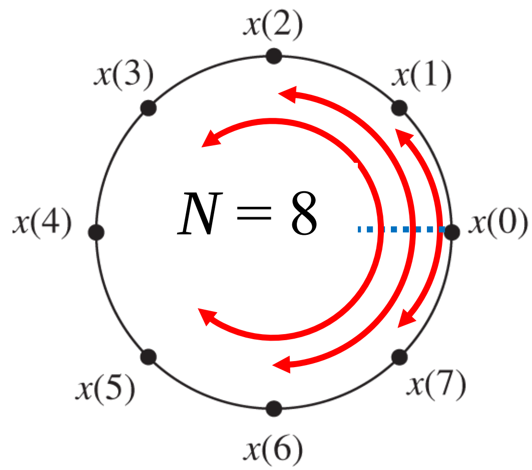
Circularly Even Sequence: Even symmetry about 0 ***on the circle***

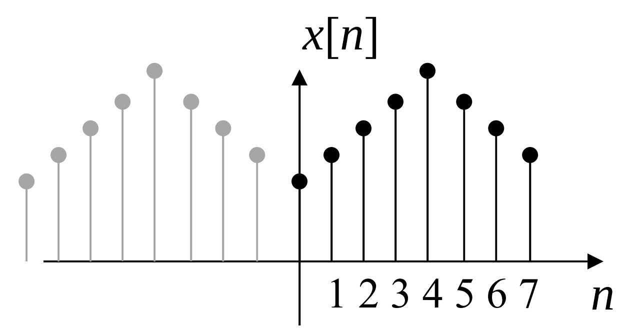
$$x[n] = x[N-n], \quad 1 \le n \le N-1$$

Not 0!!

$$x_p[n] = x_p[-n] = x_p[N-n]$$

$$\begin{aligned} x[1] &= x[8-1] \\ &= x[7] \\ x[2] &= x[8-2] \\ &= x[6] \\ x[3] &= x[8-3] \\ &= x[5] \\ x[4] &= x[8-4] \\ &= x[4] \end{aligned}$$





For complex signals – "Conjugate Even" signals have: $x_p[n] = x_p^*[N-n]$

<u>Circularly Odd Sequence</u>: Odd symmetry about 0 ***<u>on the circle</u>*** → $x[0] = 0$

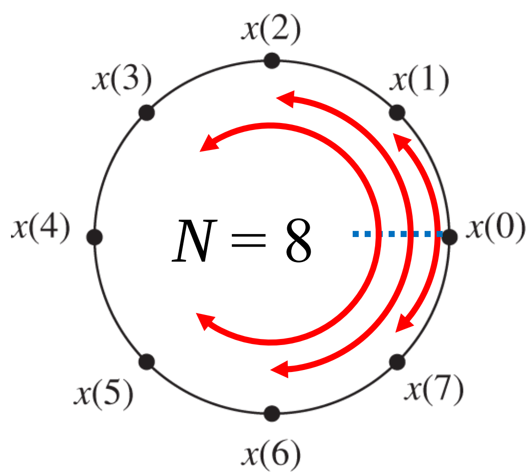
$$x[n] = -x[N-n], \quad 1 \le n \le N-1$$

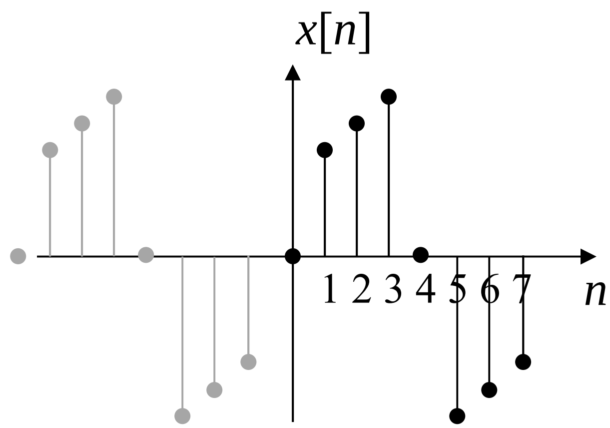
Not 0!!

$$x_p[n] = -x_p[-n] = -x_p[N-n]$$

$$\begin{aligned} x[1] &= -x[8-1] \\ &= -x[7] \\ x[2] &= -x[8-2] \\ &= -x[6] \\ x[3] &= -x[8-3] \\ &= -x[5] \\ x[4] &= -x[8-4] \\ &= -x[4] \end{aligned}$$

→ $x[4] = 0$





For complex signals – "Conjugate Odd" signals have: $x_p[n] = -x_p^*[N-n]$

Decomposing Signal into Odd & Even Parts:

Given an arbitrary signal with periodic extension $x_p[n]$ we can decompose it as

$$x_p[n] = x_{pe}[n] + x_{po}[n]$$

$$x_{pe}[n] = \frac{1}{2}\left[x_p[n] + x_p^*[N-n] \right]$$

$$x_{po}[n] = \frac{1}{2}\left[x_p[n] - x_p^*[N-n] \right]$$

Symmetries Properties of the DFT

For generality, assume that the signal (and its DFT) are complex valued.

General Results

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$

$$= \sum_{n=0}^{N-1} \left[x_R[n] + jx_I[n]\right]\left[\cos(2\pi kn/N) - j\sin(2\pi kn/N)\right]$$

$$X_R[k] = \sum_{n=0}^{N-1} \left[x_R[n]\cos(2\pi kn/N) + x_I[n]\sin(2\pi kn/N)\right]$$

$$X_I[k] = -\sum_{n=0}^{N-1} \left[x_R[n]\sin(2\pi kn/N) - x_I[n]\cos(2\pi kn/N)\right]$$

$$x_R[n] = \frac{1}{N}\sum_{k=0}^{N-1} \left[X_R[k]\cos(2\pi kn/N) - X_I[k]\sin(2\pi kn/N)\right]$$

$$x_I[n] = \frac{1}{N}\sum_{k=0}^{N-1} \left[X_R[k]\sin(2\pi kn/N) + X_I[k]\cos(2\pi kn/N)\right]$$

Real-Valued Sequences

If $x[n]$ is real valued then

$$X^{d^*}[k] = \left[\sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}\right]^* = \sum_{n=0}^{N-1} x^*[n]e^{j2\pi kn/N} = \sum_{n=0}^{N-1} x[n]e^{-j2\pi(-k)n/N} = X^d[-k]$$

But by the periodicity of the DFT we have that $X^d[-k] = X^d[N-k]$

$$X^{d^*}[k] = X^d[-k] = X^d[N-k]$$

$$X^{d^*}[0] = X^d[N]$$

Periodicity $\rightarrow X^d[0] = X^d[N]$

$X^d[0]$ must be real

$$X^{d^*}[N/2] = X^d[N/2]$$

$X^d[N/2]$ must be real

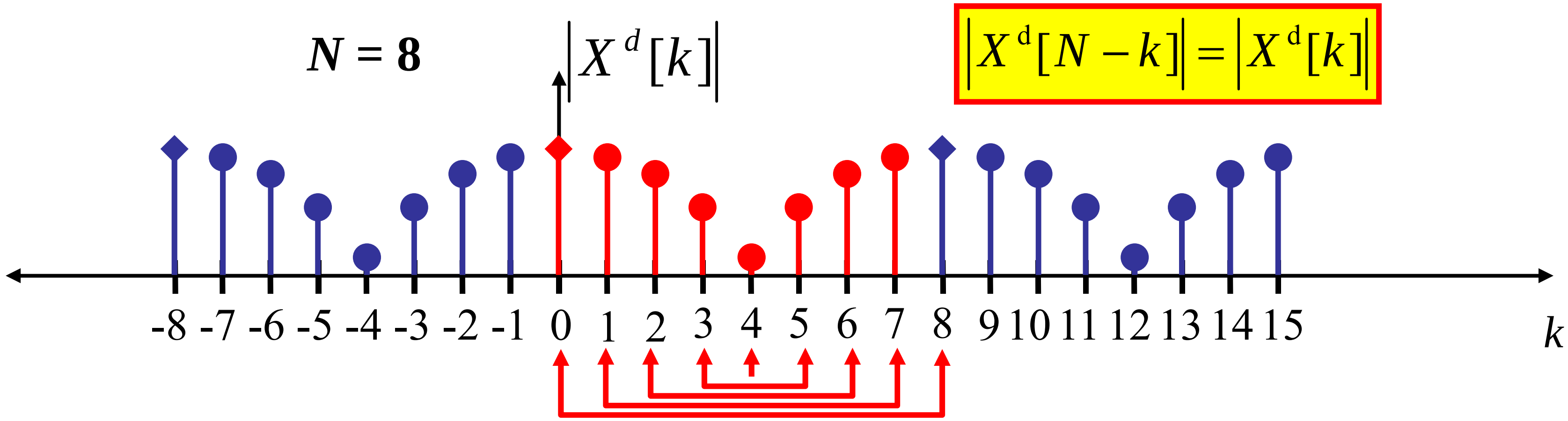
This then yields $\left|X^d[N-k]\right| = \left|X^d[k]\right| \qquad \angle X^d[N-k] = -\angle X^d[k],\ k \neq 0, N/2$

Circularly Even **"Sort of" Circularly Odd**

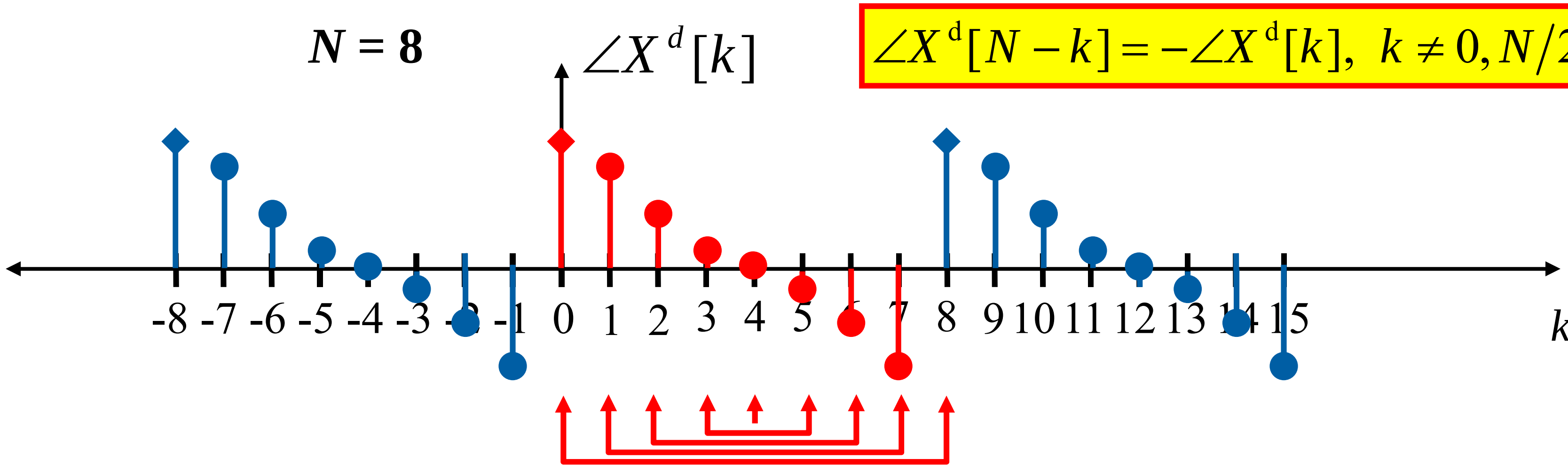
$$\angle X^d[0] = m\pi$$

$$\angle X^d[N/2] = m\pi$$

Example of DFT Symmetry for Real Signal

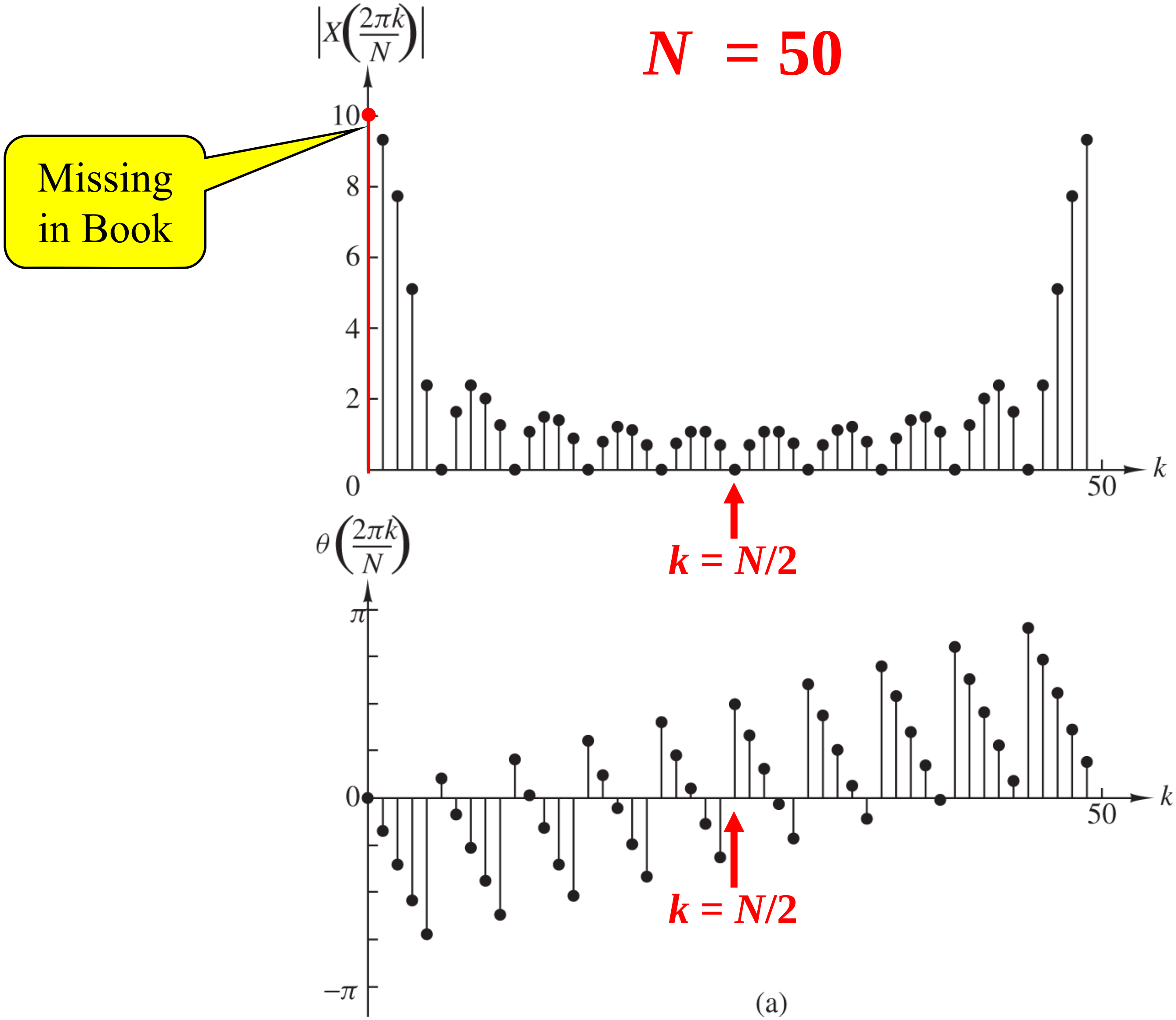


$$\left|X^{\mathrm{d}}[N-k]\right| = \left|X^{\mathrm{d}}[k]\right|$$



$$\angle X^{\mathrm{d}}[N-k] = -\angle X^{\mathrm{d}}[k], \;\; k \neq 0, N/2$$

$N = 50$



$\left|X\left(\frac{2\pi k}{N}\right)\right|$
10
8
6
4
2
0
50
k
Missing in Book
k = N/2
$\theta\left(\frac{2\pi k}{N}\right)$
π
0
50
k
$-\pi$
k = N/2

(a)

Real-Valued & Circularly Even Sequences

$x[n]$ real ⇒

$$X_R[k] = \sum_{n=0}^{N-1}\left[x_R[n]\cos(2\pi kn/N) + x_I[n]\sin(2\pi kn/N)\right]$$

$$X_I[k] = -\sum_{n=0}^{N-1}\left[x_R[n]\sin(2\pi kn/N) - x_I[n]\cos(2\pi kn/N)\right]$$

$x[n] = x[N-n]$ ⇒ = 0 (even x odd = odd)

$$X^d[k] = \sum_{n=0}^{N-1} x[n]\cos(2\pi kn/N)$$

Real & Even

$$x_R[n] = \frac{1}{N}\sum_{k=0}^{N-1}\left[X_R[k]\cos(2\pi kn/N) - X_I[k]\sin(2\pi kn/N)\right]$$

$$x_I[n] = \frac{1}{N}\sum_{k=0}^{N-1}\left[X_R[k]\sin(2\pi kn/N) + X_I[k]\cos(2\pi kn/N)\right]$$

= 0 (even x odd = odd)

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\cos(2\pi kn/N)$$

Real-Valued & Circularly Odd Sequences

$$X^{\mathrm{d}}[k] = -j\sum_{n=0}^{N-1} x[n]\sin(2\pi kn/N)$$

$$x[n] = j\frac{1}{N}\sum_{k=0}^{N-1} X^{\mathrm{d}}[k]\sin(2\pi kn/N)$$

Real & Even

See Table 7.1 for Summary of Symmetry Properties