BINGHAMTON UNIVERSITY
State University of New York

EEO 401
Digital Signal Processing

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Note Set #19

- Details of the DFT
- Reading Assignment: Sect. 7.1.2, 7.1.3, & 7.2 of Proakis & Manolakis

Definition of the DFT

So… Given ***N* signal data points** $x[n]$ for $n = 0, \ldots, N$-1
Compute N DFT points using:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N} \qquad k = 0,1,2,\ldots,N-1$$

$$\omega_k = k\frac{2\pi}{N}$$

Inverse DFT (IDFT)

So… Given N DFT points $X[k]$ for $k = 0, \ldots, N$-1
Compute N signal data points using:

$$x[n] = \frac{1}{N}\sum_{n=0}^{N-1} X[k] e^{j2\pi kn/N} \qquad n = 0,1,2,\ldots,N-1$$

DFT as a Matrix Operator (Linear Transformation)

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N} \qquad k = 0,1,2,\ldots,N-1$$

$$X[0] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi 0n/N} = x[0] + x[1] + \cdots + x[N-2] + x[N-1]$$

$$X[1] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi 1n/N} = x[0] + x[1]e^{-j2\pi 1/N} + \cdots + x[N-2]e^{-j2\pi 1(N-1)/N}$$

$$\vdots$$

$$X[N-1] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi (N-1)n/N} = x[0] + x[1]e^{-j2\pi 1(N-1)/N} + \cdots + x[N-2]e^{-j2\pi (N-1)(N-1)/N}$$

$$\underbrace{\begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix}}_{\mathbf{X}_N} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{j2\pi 1\cdot 1/N} & e^{j2\pi 2\cdot 1/N} & \cdots & e^{j2\pi (N-1)\cdot 1/N} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & e^{j2\pi 1(N-1)/N} & e^{j2\pi 2(N-1)/N} & \cdots & e^{j2\pi (N-1)(N-1)/N} \end{bmatrix}}_{\text{DFT Matrix}} \underbrace{\begin{bmatrix} x[0] \\ \mathrm{x}[1] \\ \vdots \\ \mathrm{x}[N-1] \end{bmatrix}}_{\mathbf{x}_N}$$

DFT Matrix

It is common to use the W_N symbol when discussing the DFT: $W_N \triangleq e^{-j2\pi/N}$

"N^{th} root of unity"

DFT: $$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \qquad k = 0,1,2,\ldots,N-1$$

IDFT: $$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[\text{k}] W_N^{-kn} \qquad n = 0,1,2,\ldots,N-1$$

DFT Matrix

$$\mathbf{W}_N \triangleq \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

Easy to Invert!

DFT: $$\mathbf{X}_N = \mathbf{W}_N \mathbf{x}_N$$

IDFT: $$\mathbf{x}_N = \mathbf{W}_N^{-1}\mathbf{X}_N = \frac{1}{N}\mathbf{W}_N^{*}\mathbf{X}_N$$

Properties of DFT

You've learned the properties for CTFT and for DTFT… (e.g., delay property, modulation property, convolution property, etc.) and seen that they are very similar (except having to account for the DTFT's periodicity)

Since the DFT is linked to the DTFT you'd also expect the properties of the DFT to be similar to those of the DTFT…. That is only partially true!

Linearity:

$$\left.\begin{aligned} x_1[n] &\overset{DFT}{\underset{N}{\longleftrightarrow}} X_1[k] \\ x_2[n] &\overset{DFT}{\underset{N}{\longleftrightarrow}} X_2[k] \end{aligned}\right\} \quad a_1x_1[n]+a_2x_2[n] \overset{DFT}{\underset{N}{\longleftrightarrow}} a_1X_1[k]+a_2X_2[k]$$

Periodicity:

1. DFT points computed using the DFT formula are periodic

$$X[k+N] = X[k] \quad \forall k$$

2. Signal samples **computed** using the IDFT formula are periodic

$$x[n+N] = x[n] \quad \forall n$$

Proof of #2:

$$x[n+N] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{j2\pi(n+N)k/N}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X[k] \underbrace{e^{j2\pi Nn/N}}_{=1} e^{j2\pi kn/N} = x[n]$$

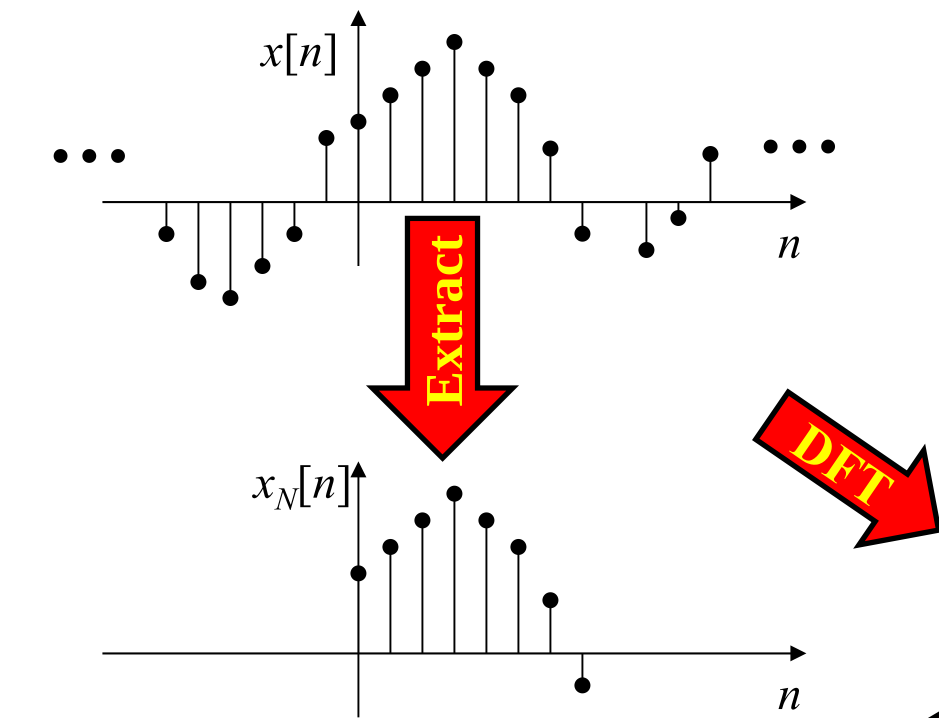
Proof of #1 is virtually identical

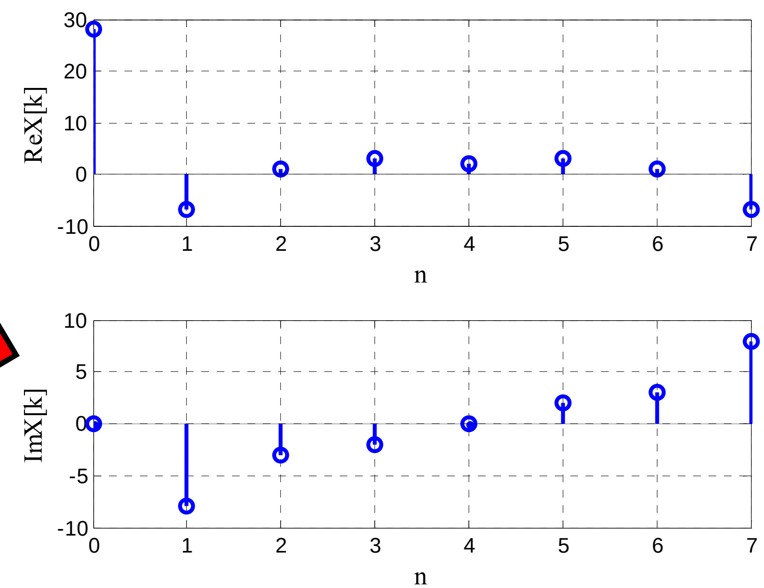
#1 is not surprising… it comes from the periodicity of the DTFT…

Note that #2 says the periodicity is for the samples AFTER doing an IDFT!!!

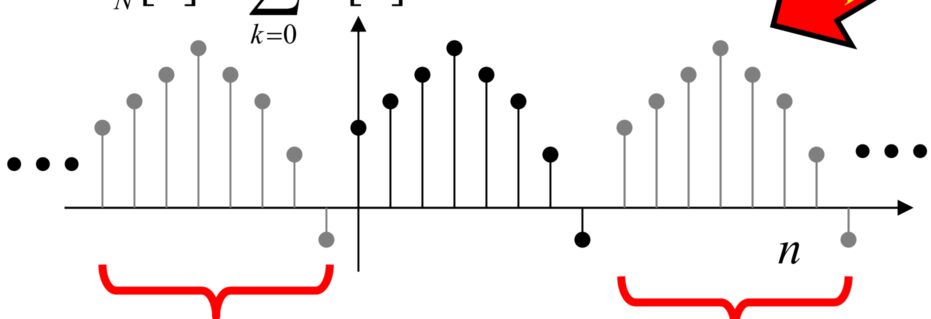
This has a big impact on other properties such as convolution & delay properties!!

Note that #2 says the periodicity is for the samples AFTER doing an IDFT!!!





$$\tilde{x}_N[n] = \sum_{k=0}^{N-1} X[k] e^{-j2\pi kn/N}$$



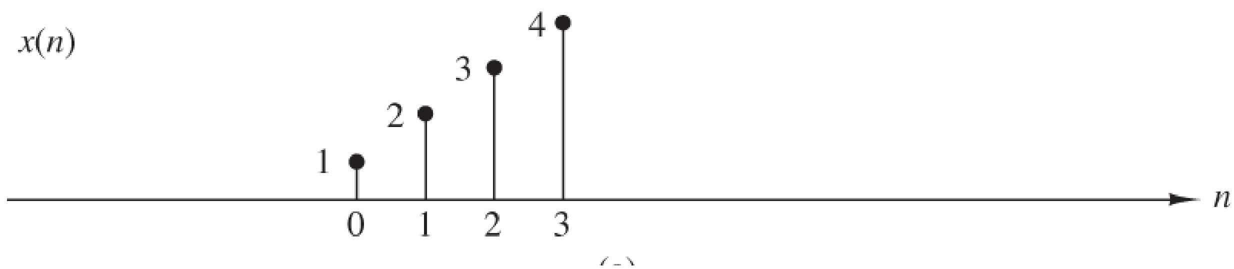
These are not "really there"… but they are "mathematically there"!!!

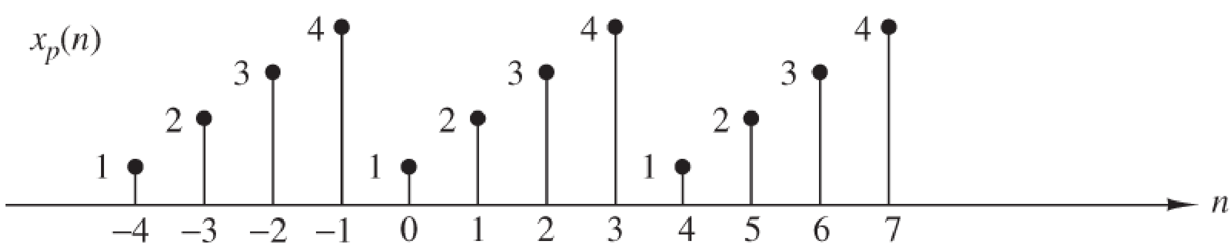
This periodic “vestige” of the signal that arises <u>in the context of the DFT</u> can be captured this way:

Let this be the L-point signal (segment) with $L \leq N$: $x[n]$

$= 0 \quad n < 0, n > L-1$

Define a N-period periodic signal by $$x_p[n] = \sum_{l=-\infty}^{\infty} x[n - lN]$$



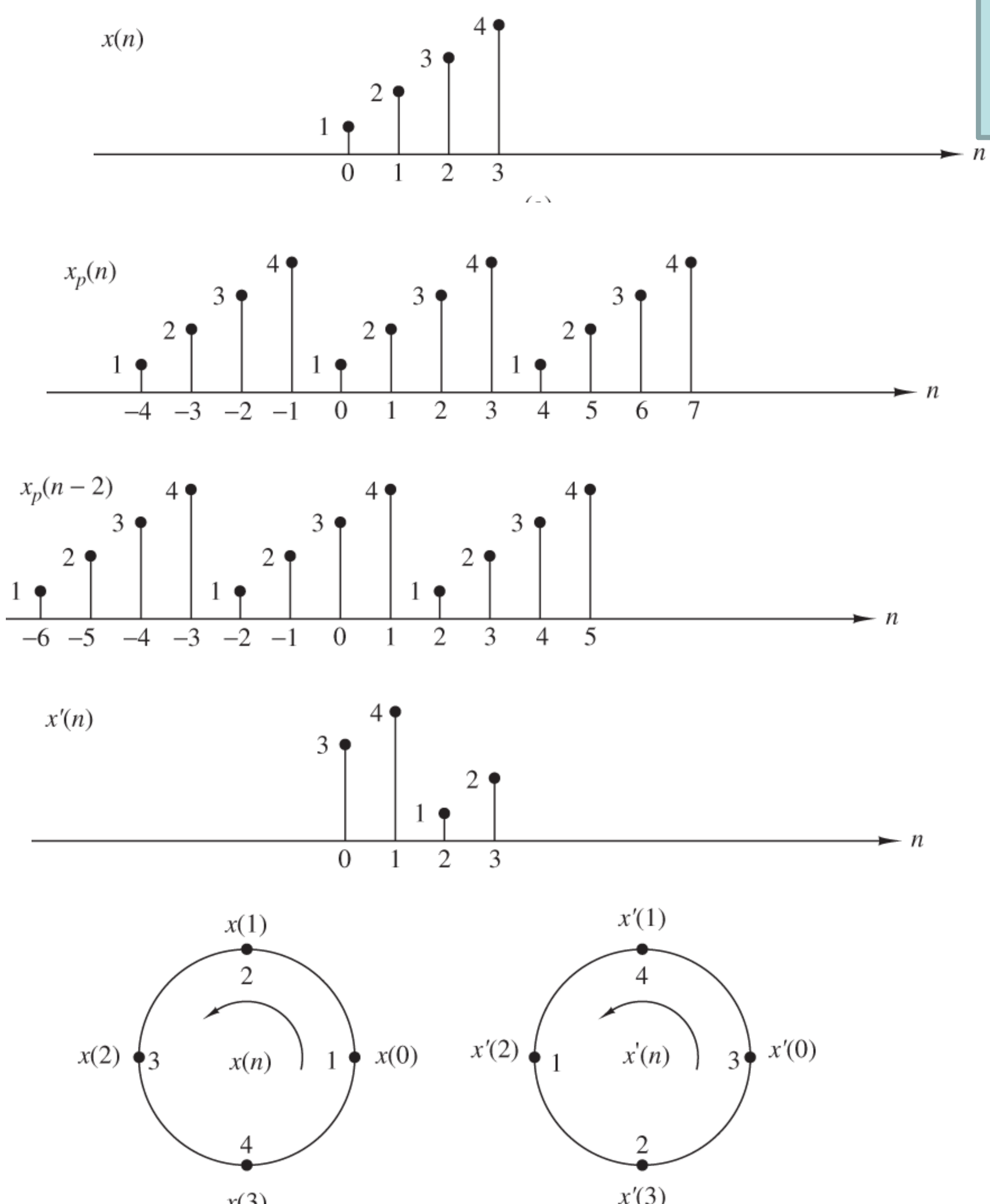


Then the periodicity property can be expressed as:

$$x_p[n] = IDFT_N\left\{DFT_N\left\{x[n]\right\}\right\}$$

N-pt DFT of signal with $L \leq N$ points implies zero-padding out to N points

Circular Shift (Defined)



(e)

Math Notation for Circular Shift

$$x[[n-k]]_N \triangleq x[n-k, \mathrm{mod}\, N]$$

n	n *mod* 4
-3	1
-2	2
-1	3
0	0
1	1
2	2
3	3
4	0
5	1
6	2
7	3
8	0

Circular Shift Property of the DFT

Recall the shift property of the DTFT (which is virtually the same as for the CTFT):

$$y[n] = x[n-l] \quad \Leftrightarrow \quad Y^{\mathrm{f}}(\omega) = e^{-j\omega l} X^{\mathrm{f}}(\omega)$$

"Regular" time shift… this is for DTFT

Imparts additional linear phase term of "integer slope"

For DFT we have a ***similar*** property but it involves ***circular*** shift rather than regular shift!!

$$y[n] = x[[n-l]]_N \quad \Leftrightarrow \quad Y^{\mathrm{d}}[k] = W_N^{-kl} X^{\mathrm{d}}[k] = e^{-j2\pi kl/N} X^{\mathrm{d}}[k]$$

Discrete frequencies @ $2\pi k/N$

This is a direct result of #2 of "Periodicity"

What this says is:

1. If you circularly shift a signal then the corresponding DFT has a linear phase term added… or alternatively
2. If you impart a linear phase shift of integer slope to the DFT, then the corresponding IDFT will have a ***circular*** shift imparted to it.

#2 is the most common scenario that arises…

Proof: The "obvious" way to "see" this is to use the periodic extension view of the IDFT:

$$x_p[n] = IDFT_N\left\{X^{\mathrm{d}}[k]\right\}$$

$$IDFT_N\left\{e^{-j2\pi kl/N}X^{\mathrm{d}}[k]\right\} = \frac{1}{N}\sum_{k=0}^{N-1}\left[e^{-j2\pi kl/N}X^{\mathrm{d}}[k]\right]e^{j2\pi kn/N}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}e^{j2\pi k(n-l)/N}X^{\mathrm{d}}[k]$$

IDFT evaluated at $n-l$... i.e. Shifted version of periodic extension

$$= x_p[n-l] = x[[n-l]]_N$$

More step-by-step proof as in the book:

When $n \geq l$ we don't need the mod operation!

$$DFT\left\{x[[n-l]]_N\right\} = \sum_{n=0}^{N-1} x[[n-l]]_N \, e^{-j2\pi kn/N}$$

$$= \sum_{n=0}^{l-1} \underbrace{x[[n-l]]_N}_{=x[N-m+n]} \, e^{-j2\pi kn/N} + \sum_{n=l}^{N-1} x[n-l] e^{-j2\pi kn/N}$$

Explicit form for mod

$$= \sum_{n=0}^{l-1} x[N-l+n] e^{-j2\pi kn/N} + \sum_{n=l}^{N-1} x[n-l] e^{-j2\pi kn/N}$$

$$= \sum_{m=N-l}^{N-1} x[m] e^{-j2\pi k(m+l)/N} + \sum_{m=0}^{N-1-l} x[m] e^{-j2\pi k(m+l)/N}$$

Change of variables in each sum

Combine into single sum

$$= e^{-j2\pi kl/N} \sum_{m=0}^{N-1} x[m] e^{-j2\pi km/N} = e^{-j2\pi kl/N} X^{d}[k]$$

Split from exp and pull out

Final Result!

Circular Modulation Property of the DFT

$$y[n] = W_N^{nm} x[n] = e^{j2\pi nm/N} x[n] \quad \Leftrightarrow \quad Y^{\mathrm{d}}[k] = X^{\mathrm{d}}[[k-m]]_N = X^{\mathrm{d}}[(k-m) \bmod N]$$

What this says is:

1. If you modulate the signal (segment) by a frequency equal to one of the discrete frequencies, then the corresponding DFT will have a ***circular*** shift imparted to it.
2. If you circularly shift a DFT then its IDFT will have a modulation imparted at a frequency equal to a discrete frequency.

The cyclic nature here is the same as for the DTFT… due to the fact that the DTFT is periodic with period of 2π.

Where this differs from the DTFT version is that the modulation frequency must be one of the discrete frequencies of the DFT.

Circular Convolution Property of the DFT

For CTFT and DTFT we had the most important property of all – the convolution property:

convolution in time domain gives multiplication in frequency domain

For the DFT this property gets changed due to the circular properties of DFT & IDFT. Later we'll see the ramifications of this.

$$\left.\begin{aligned} x_1[n] &\overset{DFT}{\underset{N}{\leftrightarrow}} X_1^d[k] \\ x_2[n] &\overset{DFT}{\underset{N}{\leftrightarrow}} X_2^d[k] \end{aligned}\right\} \Rightarrow x_1[n] \underset{N}{©} x_2[n] \overset{DFT}{\underset{N}{\leftrightarrow}} X_1^d[k] X_2^d[k]$$

My symbol for "circular" convolution of two length N signal segments… book uses a different symbol that I could not make!

Circular convolution itself is not really something we "want"… rather we end up here because we ask this question:

Given that multiplying 2 D<u>T</u>FTs corresponds to time-domain convolution…

Does the same thing hold for multiplying two DFTs???

The answer is: "sort of" but it gives circular convolution. And since LTI systems do "regular" convolution this result <u>at first</u> seems not that useful.

Proof: The periodic extension view of the IDFT provides *some* insight that this property likely holds… but we need to prove it!

You'll notice that the proof follows the line of the question we just asked: What happens in the time-domain when I multiply two DFTs???

For $m = 1, 2$

$$X_m^{d}[k] = \sum_{n=0}^{N-1} x_m[n] e^{-j2\pi nk/N}, \quad k = 0,1,2,\ldots,N-1$$

$$X_3^{d}[k] = X_1^{d}[k] X_2^{d}[k], \quad k = 0,1,2,\ldots,N-1$$

VERY important to sub in w/ different summation dummy variables!!!

$$x_3[n] = \frac{1}{N}\sum_{k=0}^{N-1} X_3^{d}[k] e^{j2\pi nk/N} = \frac{1}{N}\sum_{k=0}^{N-1} X_1^{d}[k] X_2^{d}[k] e^{j2\pi nk/N}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{m=0}^{N-1} x_1[m] e^{-j2\pi mk/N}\right]\left[\sum_{l=0}^{N-1} x_2[l] e^{-j2\pi lk/N}\right] e^{j2\pi nk/N}$$

$$= \frac{1}{N}\underbrace{\sum_{m=0}^{N-1} x_1[m]\underbrace{\sum_{l=0}^{N-1} x_2[l]\underbrace{\sum_{k=0}^{N-1} e^{j2\pi k(n-m-l)/N}}}}$$

Need to evaluate!

Aside: $\sum_{k=0}^{N-1} e^{j2\pi kn/N} = N\delta[[n]]_N = \begin{cases} N, & n = kN \ (k \text{ is integer}) \\ 0, & \text{otherwise} \end{cases}$

The result is obvious for $n = kN$ since for that case we are summing N 1s.

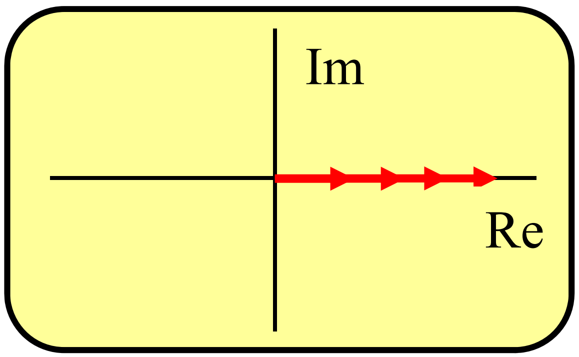
For "otherwise" this can be established via the geometric summation result:

$$\sum_{k=0}^{N-1} e^{j2\pi kn/N} = \sum_{k=0}^{N-1}\left(e^{j2\pi n/N}\right)^k = \frac{1-\left(e^{j2\pi n/N}\right)^N}{1-\left(e^{j2\pi n/N}\right)}$$

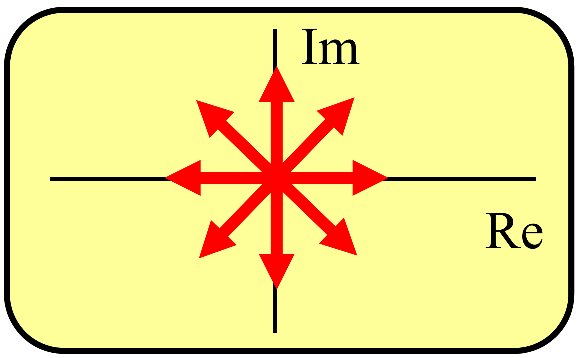
$$= \frac{1-\left(e^{j2\pi nN/N}\right)}{1-\left(e^{j2\pi n/N}\right)} = \frac{1-1}{1-\left(e^{j2\pi n/N}\right)} = 0$$

This can also be seen graphically:

For $n = kN$



For $n \neq kN$



Like evenly spaced forces that cancel out! (Yes… holds for <u>odd</u> *N* too!)

So… picking up where we left off:

$$x_3[n] = \frac{1}{N}\underbrace{\sum_{m=0}^{N-1} x_1[m] \underbrace{\sum_{l=0}^{N-1} x_2[l] \underbrace{\sum_{k=0}^{N-1} e^{j2\pi k(n-m-l)/N}}}}$$

$$= N\delta[[n-m-l]]_N$$

$$= \sum_{m=0}^{N-1} x_1[m] \underbrace{\sum_{l=0}^{N-1} x_2[l]\delta[[n-m-l]]_N}$$

Sifting Property but w/ mod N nature!

$$x_3[n] = \sum_{m=0}^{N-1} x_1[m] x_2[[n-m]]_N \triangleq x_1[n] \underset{N}{©} x_2[n]$$

Here we finish the proof & define the term "circular convolution"

"Flip & Shift" but with Mod!!! Need to understand "Circular Time Reversal" to see how this works!

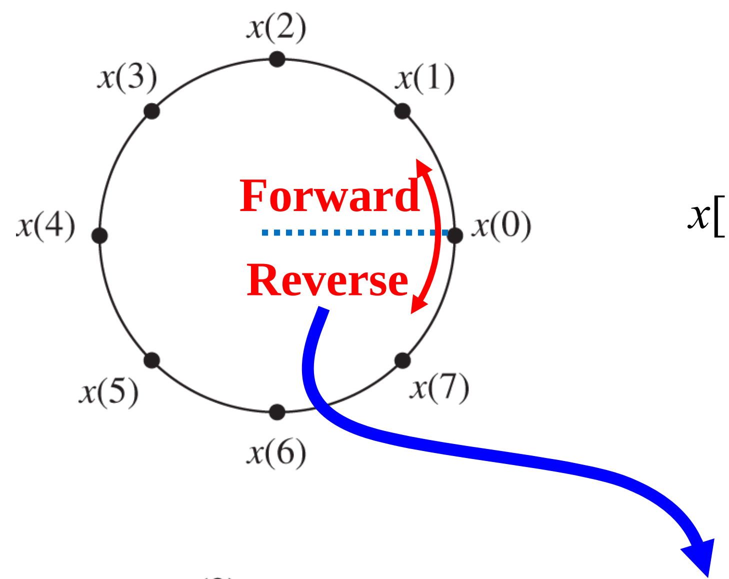
Circular Reversal: Reverse about 0 ***on the circle***

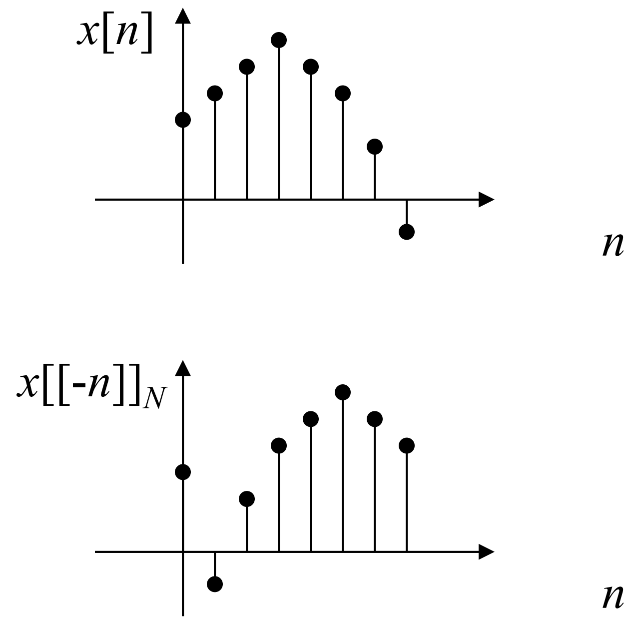
$$x[[-n]]_N = x[N-n], \quad 0 \le n \le N-1$$

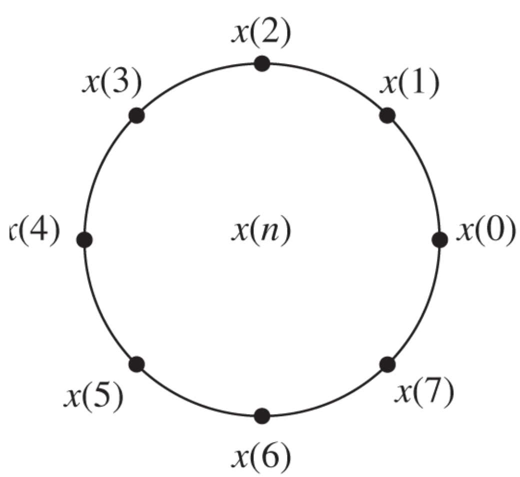
Yes 0!!

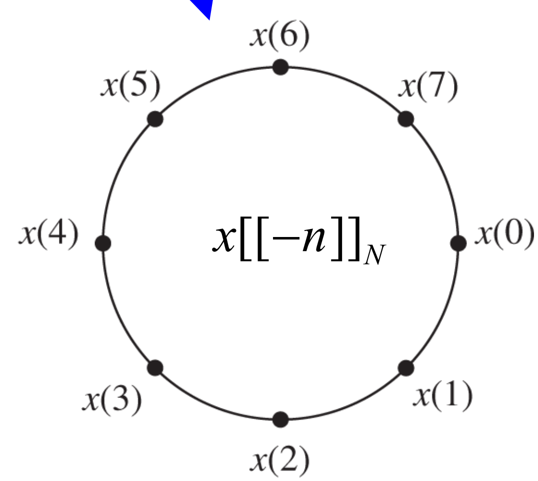
$N = 8$

$x[n]$	$x[N-n]$
$x[0]$	$x[0]$
$x[1]$	$x[7]$
$x[2]$	$x[6]$
$x[3]$	$x[5]$
$x[4]$	$x[4]$
$x[5]$	$x[3]$
$x[6]$	$x[2]$
$x[7]$	$x[1]$



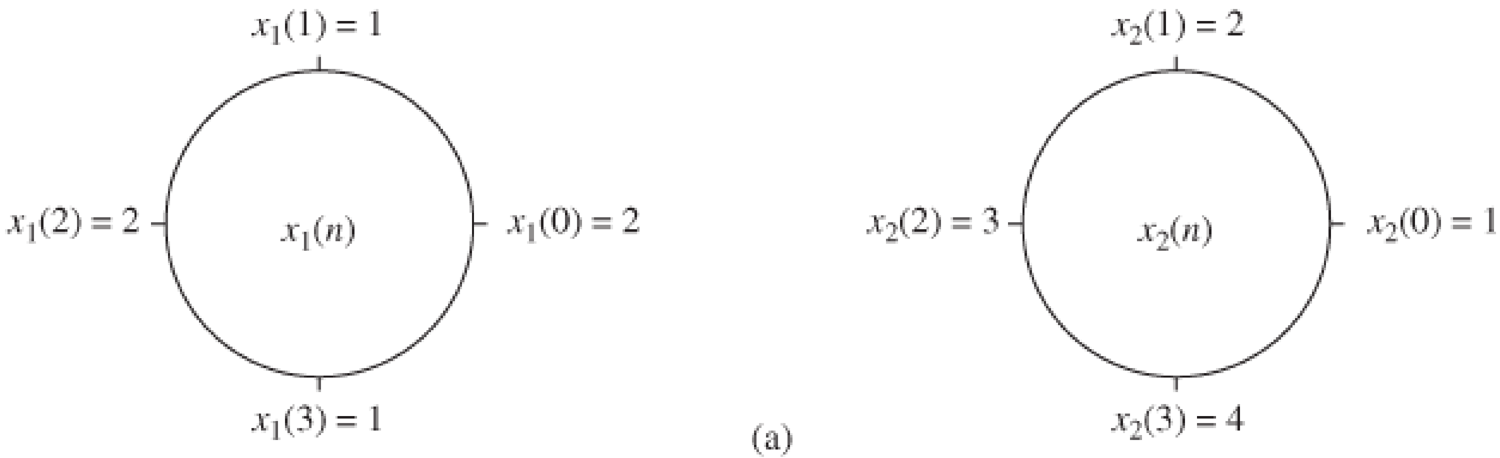






Circular Convolution Example:

$$x_3[n] = \sum_{m=0}^{N-1} x_1[m] x_2[[n-m]]_N$$



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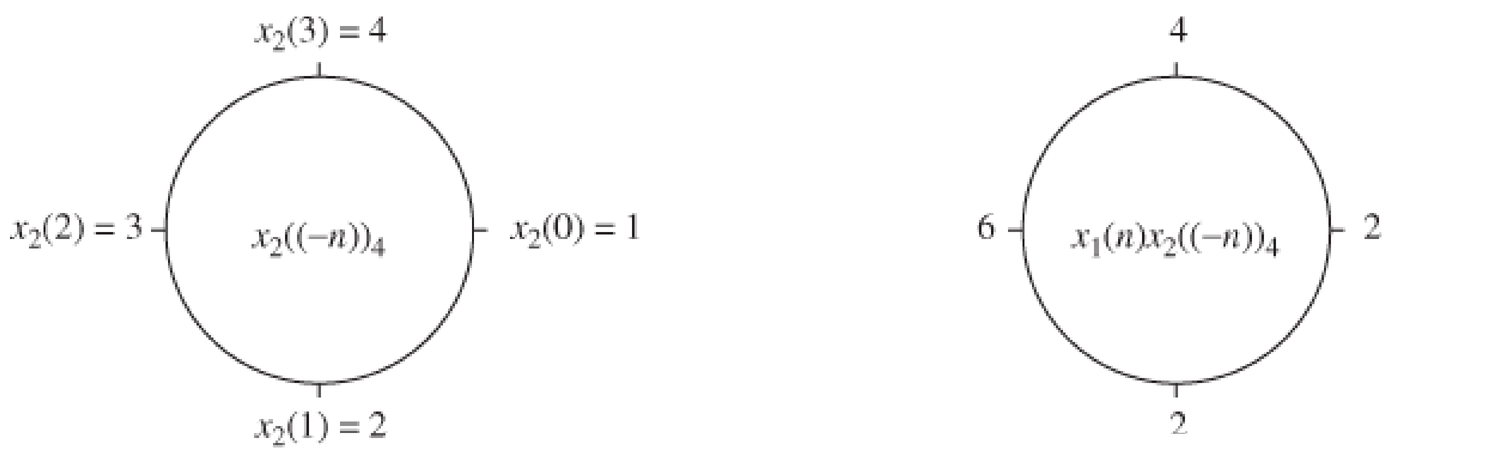


~~Folded~~ sequence
Flipped

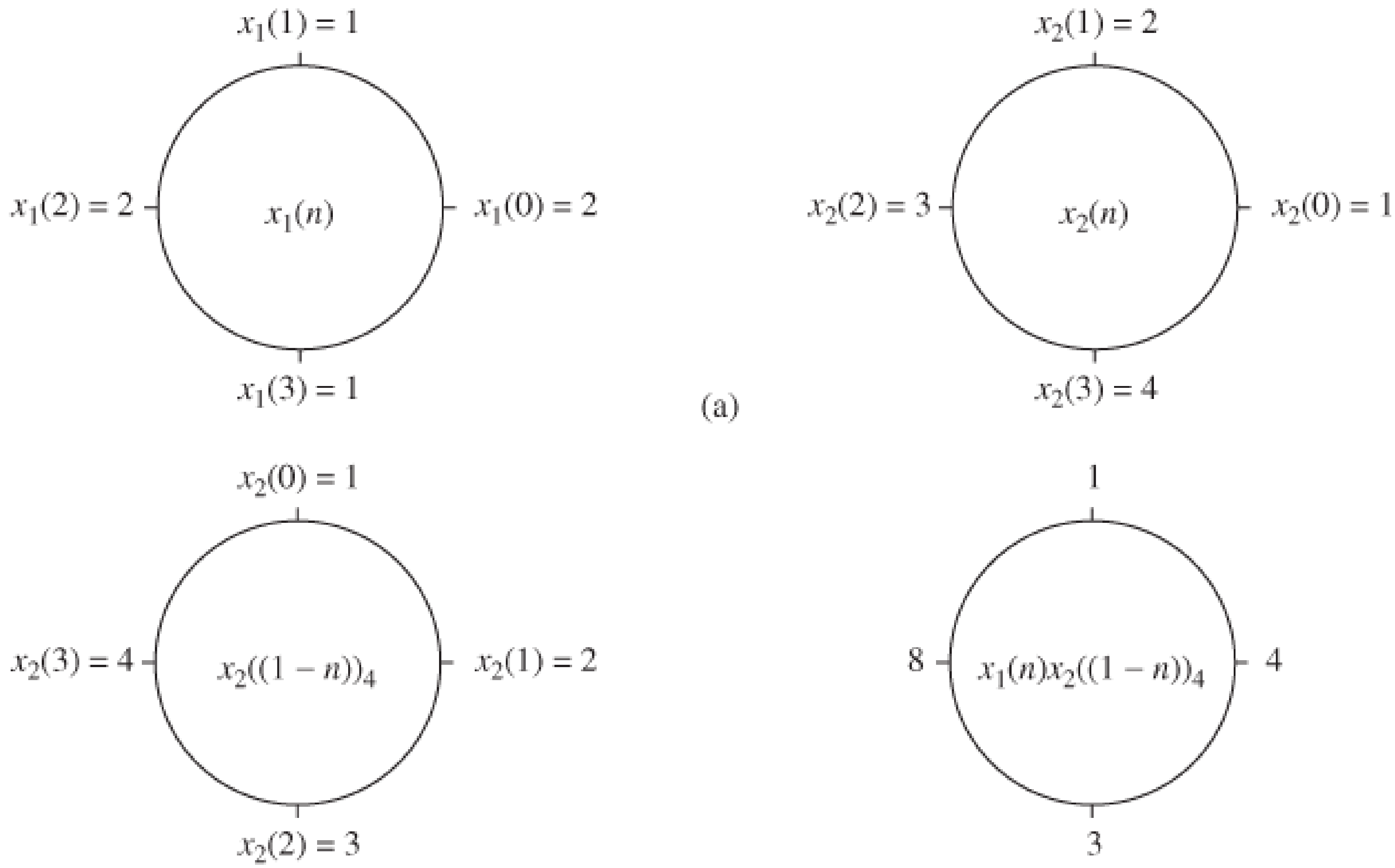
(b)

Product Sequence for $n = 0$

$x_3[0]$ = sum of these
= 2 + 4 + 6 + 2 = 14

Circular Convolution Ex. (p. 2):

$$x_3[n] = \sum_{m=0}^{N-1} x_1[m]\,x_2[[n-m]]_N$$



~~Folded~~ Flipped sequence rotated by one unit in time for $n = 1$

(c)

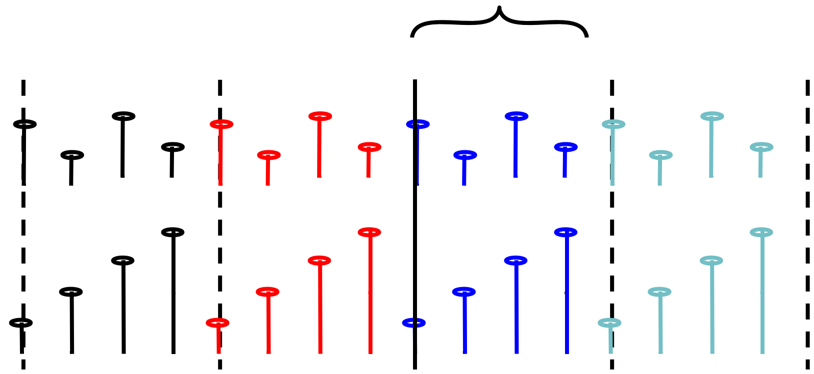
Product Sequence for $n = 1$

$x_3[1]$ = sum of these
= 4 + 1 + 8 + 3 = 16

Etc…. See textbook for the rest of the example

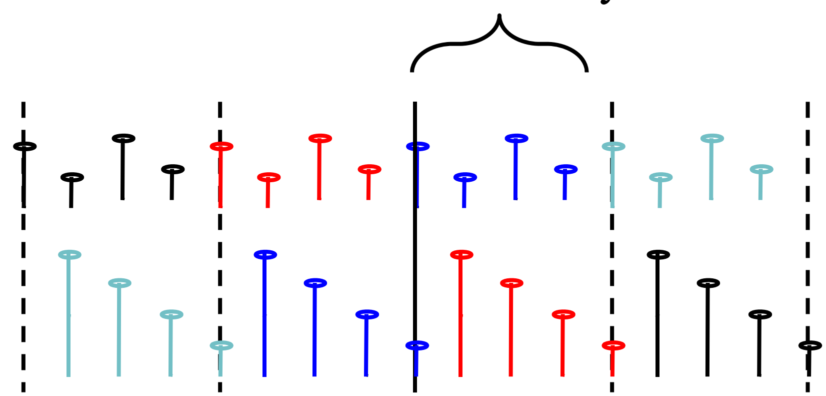
Circular Convolution Ex. (p. 3): Alternate view using periodized signals

"Original" Signals:



$n = 0$ Output Sample:

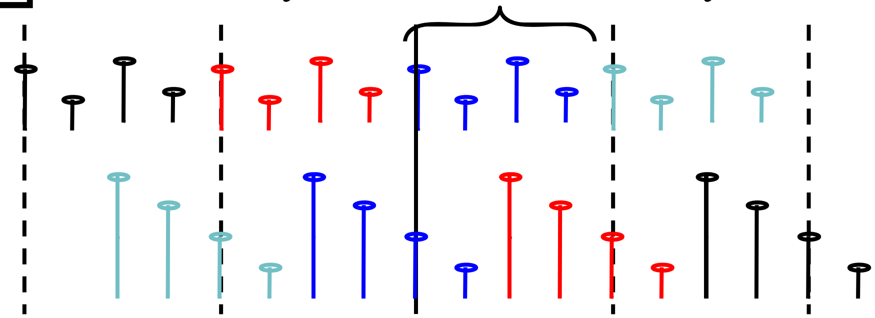
1. Flip periodized version around this point
2. No shift needed to get $n = 0$ Output Value
3. Sum over one cycle



Circular Convolution Ex. (p. 4):

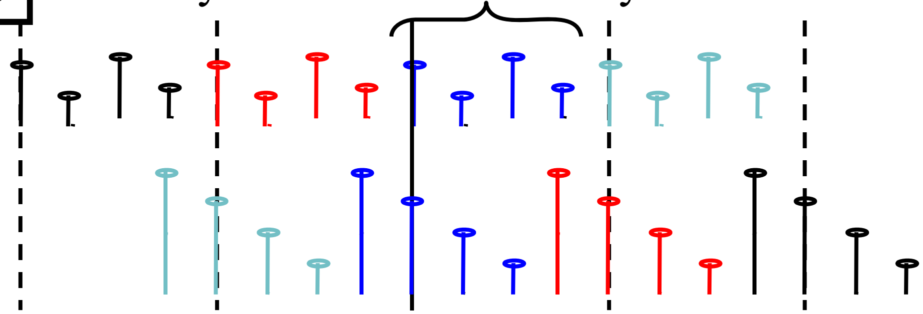
$n = 1$ Output Sample:

Shift by 1 & Sum over one cycle



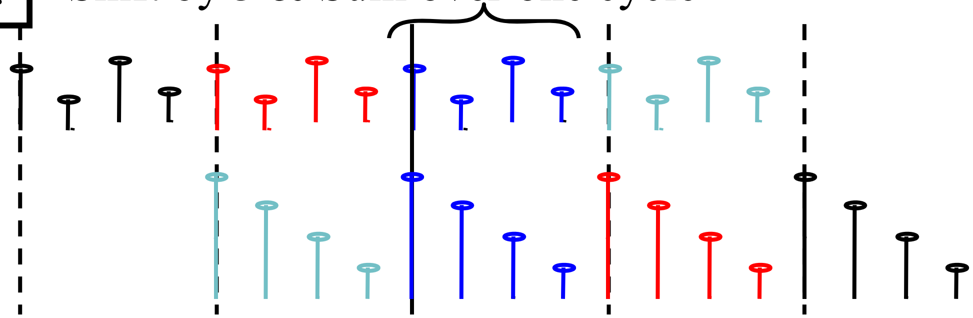
$n = 2$ Output Sample:

Shift by 2 & Sum over one cycle



$n = 3$ Output Sample:

Shift by 3 & Sum over one cycle



DFT of Product of Two Signals

$$\left.\begin{array}{l} x_1[n] \overset{DFT}{\underset{N}{\leftrightarrow}} X_1^{\mathrm{d}}[k] \\ x_2[n] \overset{DFT}{\underset{N}{\leftrightarrow}} X_2^{\mathrm{d}}[k] \end{array}\right\} \Rightarrow x_1[n]x_2[n] \overset{DFT}{\underset{N}{\leftrightarrow}} \frac{1}{N} X_1^{\mathrm{d}}[k] \underset{N}{\copyright} X_2^{\mathrm{d}}[k]$$

This is the "dual" of the Convolution Property of DFTs… so the proof is very similar.

Parseval's Theorem for DFT

$$\left.\begin{array}{l} x_1[n] \overset{DFT}{\underset{N}{\leftrightarrow}} X_1^{\mathrm{d}}[k] \\ x_2[n] \overset{DFT}{\underset{N}{\leftrightarrow}} X_2^{\mathrm{d}}[k] \end{array}\right\} \Rightarrow \sum_{n=0}^{N-1} x_1[n]x_2^*[n] = \frac{1}{N}\sum_{n=0}^{N-1} X_1^{\mathrm{d}}[k] X_2^{\mathrm{d}^*}[k]$$

Special Case:

$$\sum_{n=0}^{N-1} \left| x[n] \right|^2 = \frac{1}{N}\sum_{k=0}^{N-1} \left| X^{\mathrm{d}}[k] \right|^2$$

DFT of Complex-Conjugate

$$x[n] \overset{DFT}{\underset{N}{\longleftrightarrow}} X^{d}[k] \Rightarrow x^{*}[n] \overset{DFT}{\underset{N}{\longleftrightarrow}} X^{d^{*}}[[-k]]_{N} = X^{d^{*}}[N-k]$$

Take conjugate here

IDFT of Complex-Conjugate

$$x[n] \overset{DFT}{\underset{N}{\longleftrightarrow}} X^{d}[k] \Rightarrow x^{*}[[-n]]_{N} = x^{*}[N-n] \overset{DFT}{\underset{N}{\longleftrightarrow}} X^{d^{*}}[k]$$

Take conjugate here