BINGHAMTON
U N I V E R S I T Y
State University of New York

EEO 401
Digital Signal Processing
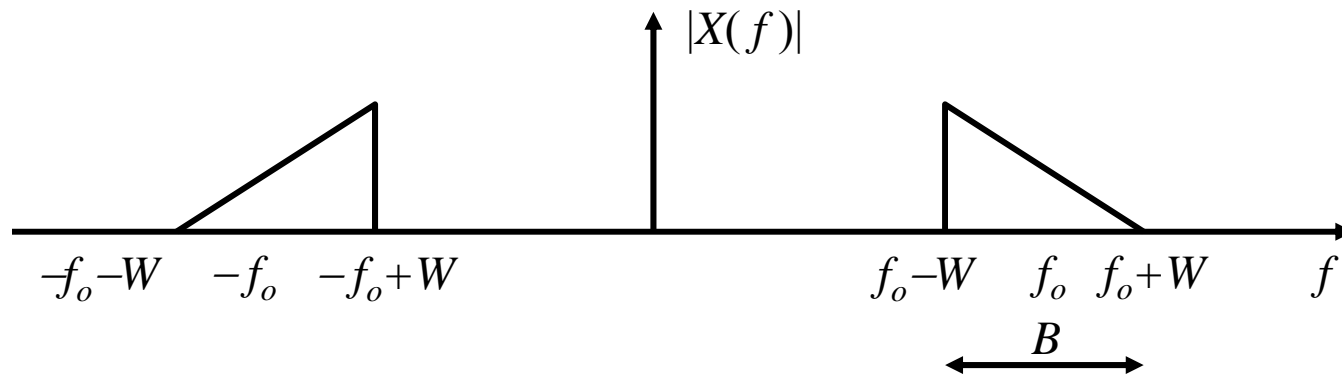## Prof. Mark Fowler

Note Set #15

- Bandpass Signal Representations
- Reading Assignment: Sect. 6.4.3 of Proakis & Manolakis

Definition: Bandpass Signal

A <u>Bandpass Signal</u> is a signal $x(t)$ whose Fourier transform $X(f)$ is nonzero only in some small band around some "central" frequency f_o.

For example:

$$X(f) = 0 \quad \text{for} \quad |f - f_o| > W \quad \text{where} \quad W < f_o.$$
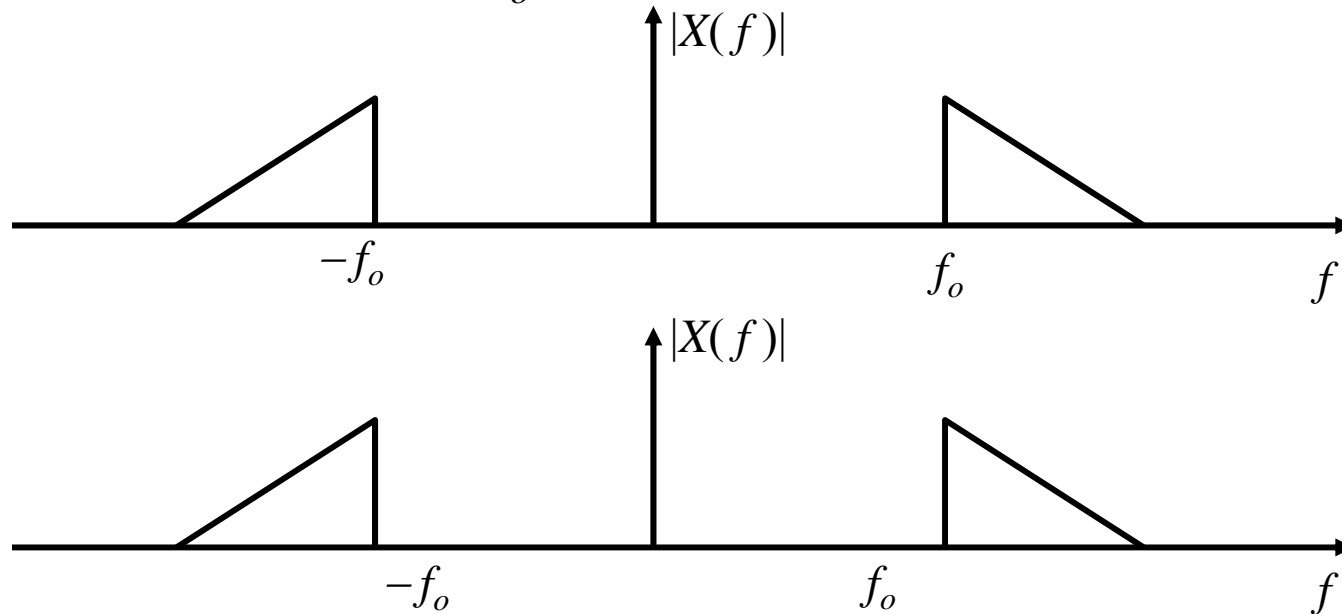

The **<u>bandwidth</u>** B of the bandpass signal = the width of the <u>positive</u>-frequency interval on which the signal is nonzero.

(Note: this is consistent with the bandwidth definition for lowpass signals).

Definition: Bandpass Signal (cont.)

Note that the choice of f_o is arbitrary:

Bandpass signals are encountered when receiving radio frequency (RF) signals such as communication and radar signals.
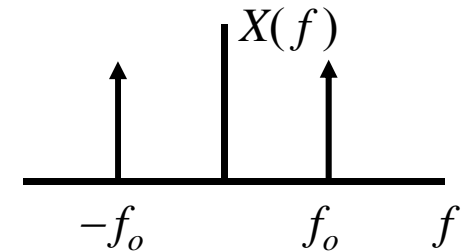
In the analysis and actual processing of BP signals it is convenient to work with a related, equivalent signal called the Equivalent Lowpass Signal. This is a natural generalization of the idea of phasor used in sophomore-level circuits.

Recall: Phasor Idea Used in Circuits

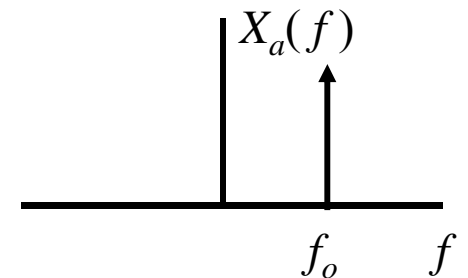Idea: Replace $A\cos(2\pi f_o t + \theta)$ by complex DC value $Ae^{j\theta}$

First, the **sinusoid**

$$x(t) = A\cos(2\pi f_o t + \theta)$$

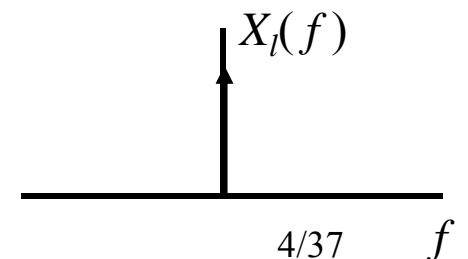$$= \frac{A}{2}e^{j(2\pi f_o t + \theta)} + \frac{A}{2}e^{-j(2\pi f_o t + \theta)}$$

gets represented by a complex-valued signal
called the **analytic signal**:

$$x_a(t) = A\,exp\{\,j(2\pi f_o t + \theta)\}$$
$$= A\cos(2\pi f_o t + \theta) + j\,A\sin(2\pi f_o t + \theta)$$

Then to get the **phasor**, we frequency-shift the
analytic signal down by f_o to get:

$$x_l = exp\{-j2\pi f_o t\}\,x_a(t)$$
$$= Ae^{j\theta}$$

4/37

Recall: Phasor Idea Used in Circuits (cont.)

phasor = equivalent *lowpass* signal representing the sinusoid
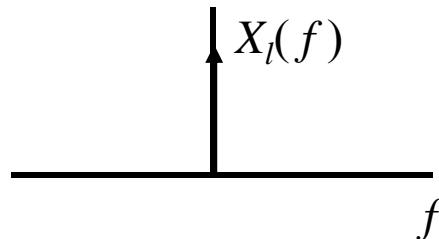(that's why we used the subscript l – for lowpass).

Note that this equivalent lowpass signal is complex valued,
whereas the bandpass signal (the sinusoid) it represents is
real valued.

Alternate View – Frequency Domain:

1. Suppress the negative frequency part of the sinusoid:

$X_a(f)$

f_o f
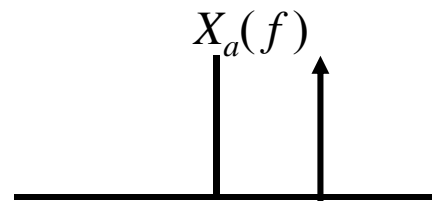
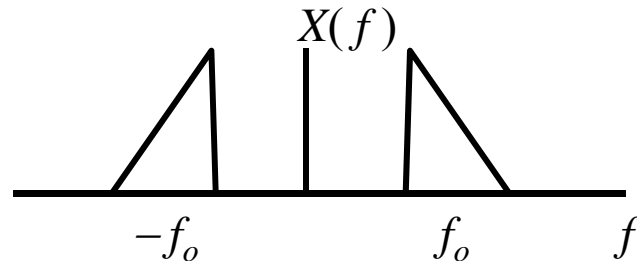2. Frequency-shift the positive frequency part down to DC:

$X_l(f)$

f

Frequency-Domain View of Equiv. LP Signal

Now… use this FD view to do the same thing for a general bandpass signal that consists of more than one frequency.

   ….Then after that we interpret the results in the time domain.

Bandpass Signal's Fourier Transform:

$X(f)$

$-f_o$ f_o f

Now to get the FT of the so-called <u>Analytic Signal</u> we suppress the negative frequencies:

$X_a(f)$

f_o f
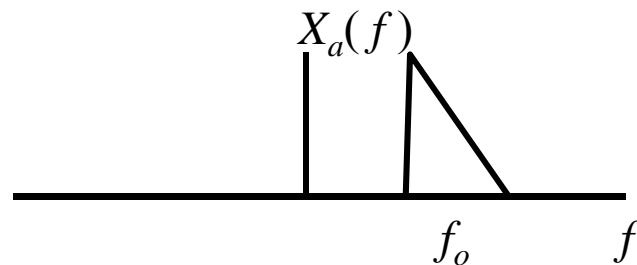
<u>Note</u>: Since $|X_a(f)|$ is NOT even-symmetric, the TD signal $x_a(t)$ is complex-valued. (see Porat p. 12, #9)

F-D View of Equiv. LP Signal (cont.)

System View of Generating Analytic Signal: define a system frequency response $H(f)$ such that

$$H(f) = \begin{cases} -j & f > 0 \\ 0 & f = 0 \\ +j & f < 0 \end{cases} \qquad \text{then}$$

$$\boxed{\begin{aligned} X_a(f) &= X(f) + jH(f)X(f) \\ &= X(f) + j\hat{X}(f) \end{aligned}}$$

where $\qquad \hat{X}(f) = H(f)X(f)$
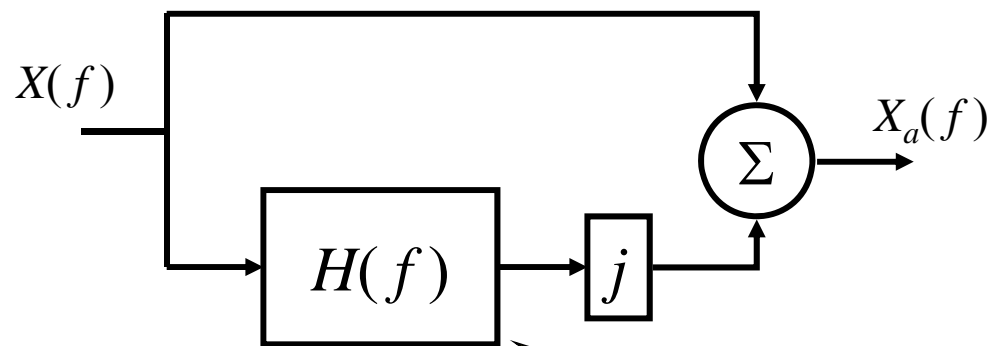
Called Hilbert Transformer

F-D View of Equiv. LP Signal (cont.)

Then to get the **FT of the Equivalent Lowpass Signal**, frequency-shift the analytic signal down by f_o to get:

$X_l(f)$

f
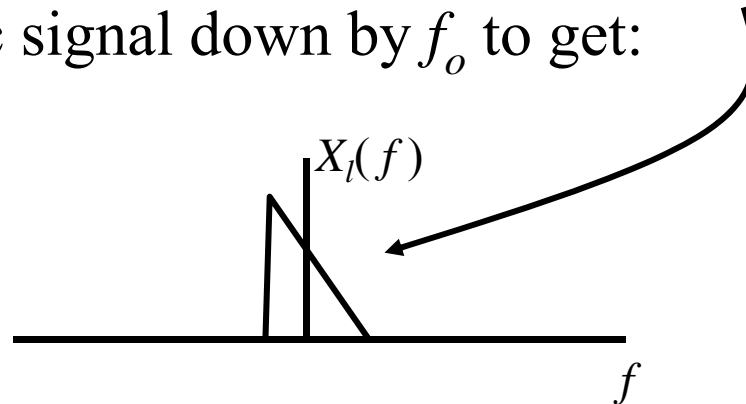
Also called:
Equivalent
Baseband Signal

Note that because $|X_l(f)|$ does not necessarily have even symmetry, the underline equivalent lowpass signal is complex valued,
….. whereas the bandpass signal it represents is real valued.

Now… how do we describe the ELP signal in the Time-Domain?

T-D View of ELP Signal

Consider the IFT of $H(f)X(f)$: $\hat{x}(t) = \mathscr{F}^{-1}\{\hat{X}(f)\} = \mathscr{F}^{-1}\{H(f)X(f)\}$

$$x_a(t) = x(t) + j\hat{x}(t) \qquad (\blacktriangle)$$

Let $x_l(t)$ be the time-domain signal that corresponds to $X_l(f)$. Because it is the frequency-shifted version of $x_a(t)$ …. using the frequency-shift property of FT gives:

$$x_l(t) = e^{-j2\pi f_o t} x_a(t) \qquad (\bigstar)$$

(Note: this is the same as an equation above for the phasor case!)

This, can then be written as:

$$x_l(t) = e^{-j2\pi f_o t}\left[x(t) + j\hat{x}(t)\right]$$

I&Q Form of ELP Signal

An <u>extremely</u> useful viewpoint for the ELP signal is the I&Q form:

… since $x_l(t)$ is complex-valued (see comment above in frequency-domain discussion), we can write its real and imaginary parts, which we will denote as

$$x_l(t) = x_i(t) + jx_q(t) \qquad (\dagger)$$

where subscripts i and q are for In-phase (I) and Quadrature (Q).

We'd now like to find relationships between the bandpass signal $x(t)$ and the I-Q components of the lowpass equivalent signal.

Relationship: I&Q Parts and BP Signal

Solving (✱) for the analytic signal gives

$$x_a(t) = e^{j2\pi f_o t} x_l(t) \qquad (\blacksquare)$$

(Makes sense… $x_a(t)$ is $x_l(t)$ shifted up.)

Using the I-Q form given in (✝) gives:

$$x_a(t) = e^{j2\pi f_o t}\left[x_i(t) + jx_q(t)\right]$$

$$= \left[\cos(2\pi f_o t) + j\sin(2\pi f_o t)\right]\left[x_i(t) + jx_q(t)\right]$$

$$= \left[x_i(t)\cos(2\pi f_o t) - x_q(t)\sin(2\pi f_o t)\right]$$

$$\qquad + j\left[x_i(t)\sin(2\pi f_o t) + x_q(t)\cos(2\pi f_o t)\right]$$

By (▲)

$$= x(t) + j\hat{x}(t)$$

Relationship: I&Q Parts and BP Signal (cont.)

This shows how the I&Q components are related to the BP signal:

$$x(t) = x_i(t)\cos(2\pi f_o t) - x_q(t)\sin(2\pi f_o t) \qquad (\triangle)$$

Similarly – but less important – we have:

$$\hat{x}(t) = x_i(t)\sin(2\pi f_o t) + x_q(t)\cos(2\pi f_o t)$$

Envelope/Phase Form of ELP Signal

This is an alternate form (but equally important to IQ form) of the ELP signal. Note in (**†**) that the I&Q form is a "rectangular form" for the complex ELP signal.

So… converting to a "polar form" gives:

$$x_l(t) = A(t)e^{j\theta(t)}$$

(✻)

where…

$$A(t) = \sqrt{x_i^2(t) + x_q^2(t)} \geq 0$$

Note Similarity to Phasor!!

But…
Time-<u>varying</u>
Envelope &
Magnitude

$$\theta(t) = \arctan\left\{\frac{x_q(t)}{x_i(t)}\right\}$$

Relationship: Env/Phase and I&Q

Often we need to convert between the two forms (rect & polar).
If in (�֍) we expand the complex exponential:

$$x_l(t) = A(t)e^{j\theta(t)}$$

$$= \underbrace{A(t)\cos[\theta(t)]}_{x_i(t)} + j\underbrace{A(t)\sin[\theta(t)]}_{x_q(t)}$$

By (✝)

Thus….

$$\boxed{\begin{aligned} x_i(t) &= A(t)\cos[\theta(t)] \\ x_q(t) &= A(t)\sin[\theta(t)] \end{aligned}}$$

Envelope/Phase Form of BP Signal

We already saw Env/Phase form for the ELP signal…
        Do we get something similar for the original BP signal ??

Using (■) and (✿) we can write

$$x_a(t) = e^{j2\pi f_o t}[A(t)e^{j\theta(t)}]$$

$$= A(t)e^{j[2\pi f_o t + \theta(t)]}$$

$$= A(t)\cos[2\pi f_o t + \theta(t)] + jA(t)\sin[2\pi f_o t + \theta(t)]$$

$$= x(t) + j\hat{x}(t)$$

By (▲)

$$x(t) = A(t)\cos[2\pi f_o t + \theta(t)] \qquad (\clubsuit)$$

Envelope/Phase Form of BP Signal (cont.)

So what we have just shown is:

Any BP signal can be expressed as:

$$x(t) = A(t) \cos[2\pi f_o t + \theta(t)]$$

where $A(t) \geq 0$.

Note: $A(t)$ and $\theta(t)$ vary slowly compared to $\cos(2\pi f_o t)$.

The LPE signal has the same envelope and phase as the BP signal
… compare (♣) and (✼).

$$x_l(t) = A(t) e^{j\theta(t)}$$

Analog Generation of I&Q Components

As stated earlier… processing for radar & communication is actually implemented using the ELP signal.
- Thus we need some way to get the ELP signal from a received BP signal…
- The I&Q form is the most commonly used

So… given the BP signal

$$x(t) = A(t)\cos[2\pi f_o t + \theta(t)]$$

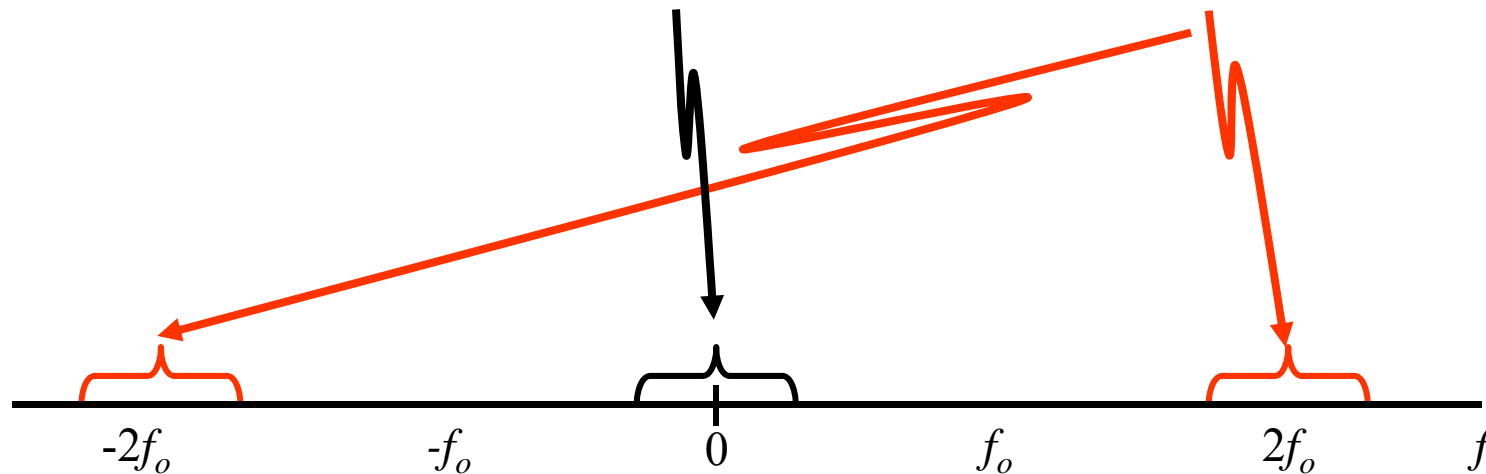we need to be able to extract through processing the I&Q signals:

$$x_i(t) = A(t)\cos[\theta(t)]$$
$$x_q(t) = A(t)\sin[\theta(t)]$$

These give the clue as to how to extract the I-Q signals by using analog techniques. Using trigonometric identities:

$$2x(t)\cos(2\pi f_o t) = 2\big[A(t)\cos(2\pi f_o t + \theta(t))\big]\cos(2\pi f_o t)$$

$$= \underbrace{A(t)\cos[\theta(t)]}_{x_i(t)} + \underbrace{A(t)\cos[2\pi(2f_o)t + \theta(t)]}_{x(t).... \text{ but centered at } 2f_o}$$

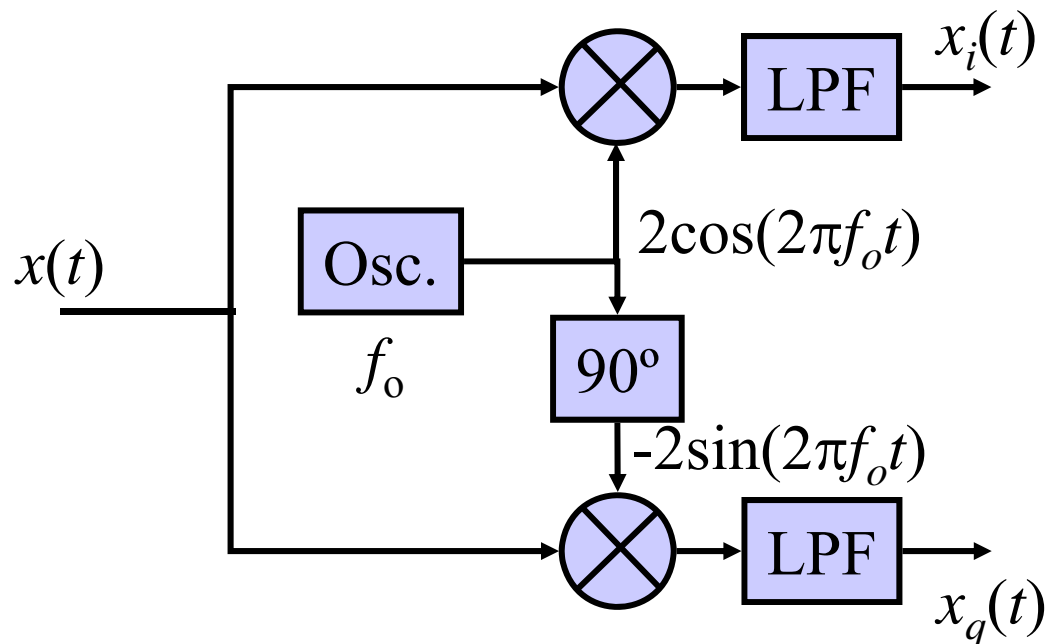|					
$-2f_o$	$-f_o$	0	f_o	$2f_o$	f

Analog Generation of I&Q Components (cont.)

Similarly, we also get….

$$2x(t)[-\sin(2\pi f_o t)] = 2\big[A(t)\cos(2\pi f_o t + \theta(t))\big][-\sin(2\pi f_o t)]$$

$$= \underbrace{A(t)\sin[\theta(t)]}_{x_q(t)} - \underbrace{A(t)\sin[2\pi(2f_o)t + \theta(t)]}_{\hat{x}(t)\dots \text{ but centered at } 2f_o}$$

Analog Circuitry to Generate I&Q

Digital Generation of I&Q Components

ANALOG DEVICES

67 MSPS Digital Receive Signal Processor

AD6620

FEATURES
High Input Sample Rate
  67 MSPS Single Channel Real
  33.5 MSPS Diversity Channel Real
  33.5 MSPS Single Channel Complex
NCO Frequency Translation
  Worst Spur Better than –100 dBc
  Tuning Resolution Better than 0.02 Hz
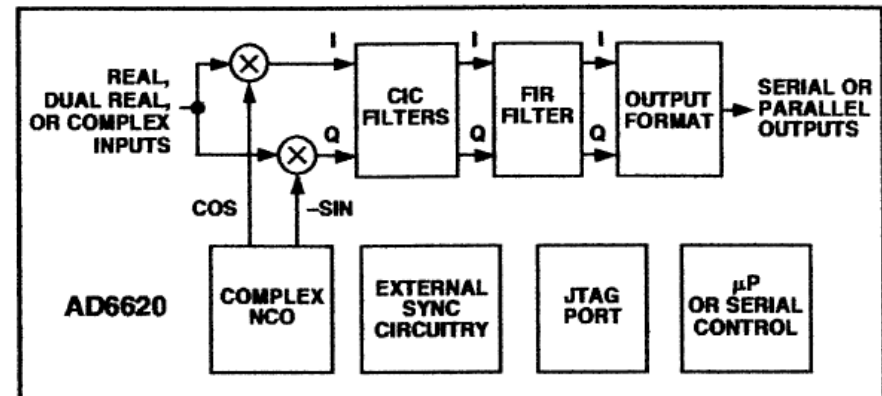**2nd Order Cascaded Integrator Comb FIR Filter**
  Linear Phase, Fixed Coefficients
  Programmable Decimation Rates: 2, 3 . . . 16
5th Order Cascaded Integrator Comb FIR Filter
  Linear Phase, Fixed Coefficients

FUNCTIONAL BLOCK DIAGRAM

Uses of These Ideas
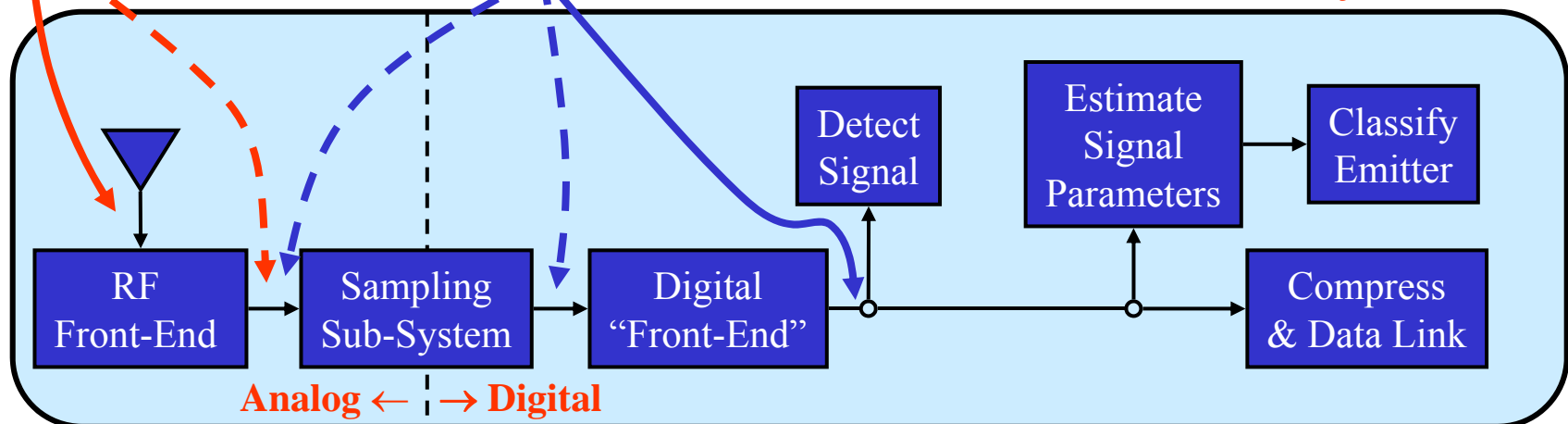
- ## Bandpass Signal Model
  - usually <u>used to *model* RF signals</u> in <u>radar and communications</u>
  - also often <u>used to *model* acoustic signals</u> in <u>sonar</u>
  - not generally used for audio/speech signals
- ## Lowpass Equivalent Signal
  - used as a <u>conceptual tool</u> to aid analysis/design
  - used as the <u>actual representation in real processing</u>

Receiver From Our Case Study

RF Front-End

Sampling Sub-System

Digital "Front-End"

Detect Signal

Estimate Signal Parameters

Classify Emitter

Compress & Data Link

Analog ← | → Digital

Uses of These Ideas (cont.)

- ## Analytic Signal
  - generally used as a <u>conceptual tool</u> to prove results
  - usually applied directly to the continuous-time RF bandpass signal
  - There are <u>occasions where we actually compute</u> the analytic signal of a real-valued digital signal,
    - but usually applying it some real-valued lowpass signal.
  - *See MATLAB Warning Below*

- ## Hilbert Transform of a Signal
  - generally used as a <u>conceptual tool</u> to prove results
  - There are <u>occasions where we actually compute</u> the Hilbert transform of a real-valued digital signal,
    - but usually applying it some real-valued lowpass signal.
  - *See MATLAB Warning Below*

MATLAB Warning

MATLAB has a command that is called "hilbert". The Help entry on MATLAB for this command is:

"HILBERT(X) is the Hilbert transform of the real part of vector X. The real part of the result is the original real data; the imaginary part is the actual Hilbert transform."

Thus, executing hilbert(x) does **NOT** return the Hilbert transform of x;

- It gives the analytic signal – see (▲).

Summary of Relationships

BP Signal & Its Hilbert Transform

$$x(t) = A(t)\cos[2\pi f_o t + \theta(t)]$$

$$\hat{x}(t) = A(t)\sin[2\pi f_o t + \theta(t)]$$

$- - - - -$

$$x(t) = \text{Re}\{x_a(t)\}$$

$$\hat{x}(t) = \text{Im}\{x_a(t)\}$$

$- - - - -$

$$x(t) = x_i(t)\cos(2\pi f_o t) - x_q(t)\sin(2\pi f_o t)$$

$$\hat{x}(t) = x_i(t)\sin(2\pi f_o t) + x_q(t)\cos(2\pi f_o t)$$

****

$$A(t) = \sqrt{x^2(t) + \hat{x}^2(t)}$$

$$\theta(t) = \arctan\left\{\frac{\hat{x}(t)}{x(t)}\right\} - 2\pi f_o t$$

Analytic Signal

$$x_a(t) = x(t) + j\hat{x}(t)$$

$$x_a(t) = x_l(t)e^{j2\pi f_o t}$$

$$x_a(t) = A(t)e^{j[2\pi f_o t + \theta(t)]}$$

****

$$A(t) = |x_a(t)|$$

$$\theta(t) = [\angle x_a(t)] - 2\pi f_o t$$

Equiv. Lowpass Model

$$x_l(t) = x_a(t)e^{-j2\pi f_o t}$$

$$x_l(t) = A(t)e^{j\theta(t)}$$

$$x_l(t) = x_i(t) + jx_q(t)$$

****

$$x_i(t) = \text{Re}\{x_l(t)\}$$

$$x_q(t) = \text{Im}\{x_l(t)\}$$

$- - - -$

$$x_i(t) = A(t)\cos[\theta(t)]$$
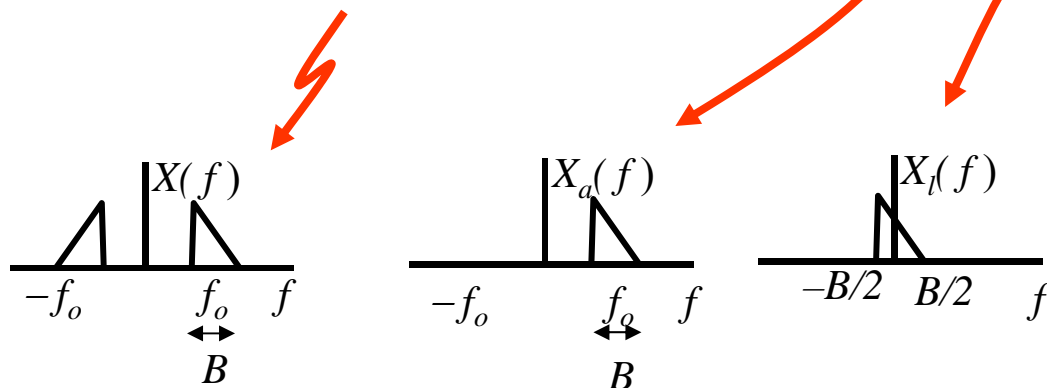
$$x_q(t) = A(t)\sin[\theta(t)]$$

$- - - -$

$$x_i(t) = x(t)\cos(2\pi f_o t) + \hat{x}(t)\sin(2\pi f_o t)$$

$$x_q(t) = \hat{x}(t)\cos(2\pi f_o t) - x(t)\sin(2\pi f_o t)$$

****

$$A(t) = |x_l(t)| = \sqrt{x_i^2(t) + x_q^2(t)}$$

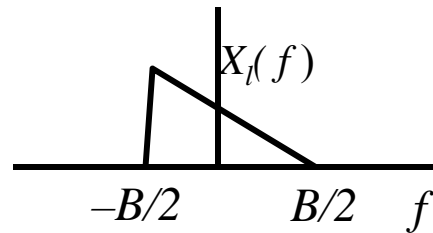$$\theta(t) = \angle x_l(t) = \arctan\left\{\frac{x_q(t)}{x_i(t)}\right\}$$

Sampling Rate Needed for ELP Signal

Given a complex-valued equivalent lowpass signal, what is an appropriate sampling rate to use?

To answer this question… look at the ELP signal's Fourier transform:

Sampling this signal is no different than sampling some real-valued lowpass signal: choose $F_s > 2f_{max}$

     …. in this case gives $F_s > 2(B/2) = B$.

Now does this make sense?

Sampling Rate Needed for ELP Signal (cont.)

Now does this make sense?

<u>Bandpass Sampling</u> on the corresponding bandpass signal (BPS)
   ….would require $F_s > 2B$,

BUT… need only half that rate for the ELP signal!!!

Do we really need only half the amount of information to represent the ELPS as we need for the BPS?

Would that even make sense? Since ELPS $\leftrightarrow$ BPS ?

It doesn't at first!!!!   ***BUT*** … the ELPS is complex
   it requires a real sample value *and*
   an imaginary sample value for *each* signal sample

$$[(I+Q) @\ F_s = B]\ =\ [BPS @\ F_s = 2B]$$

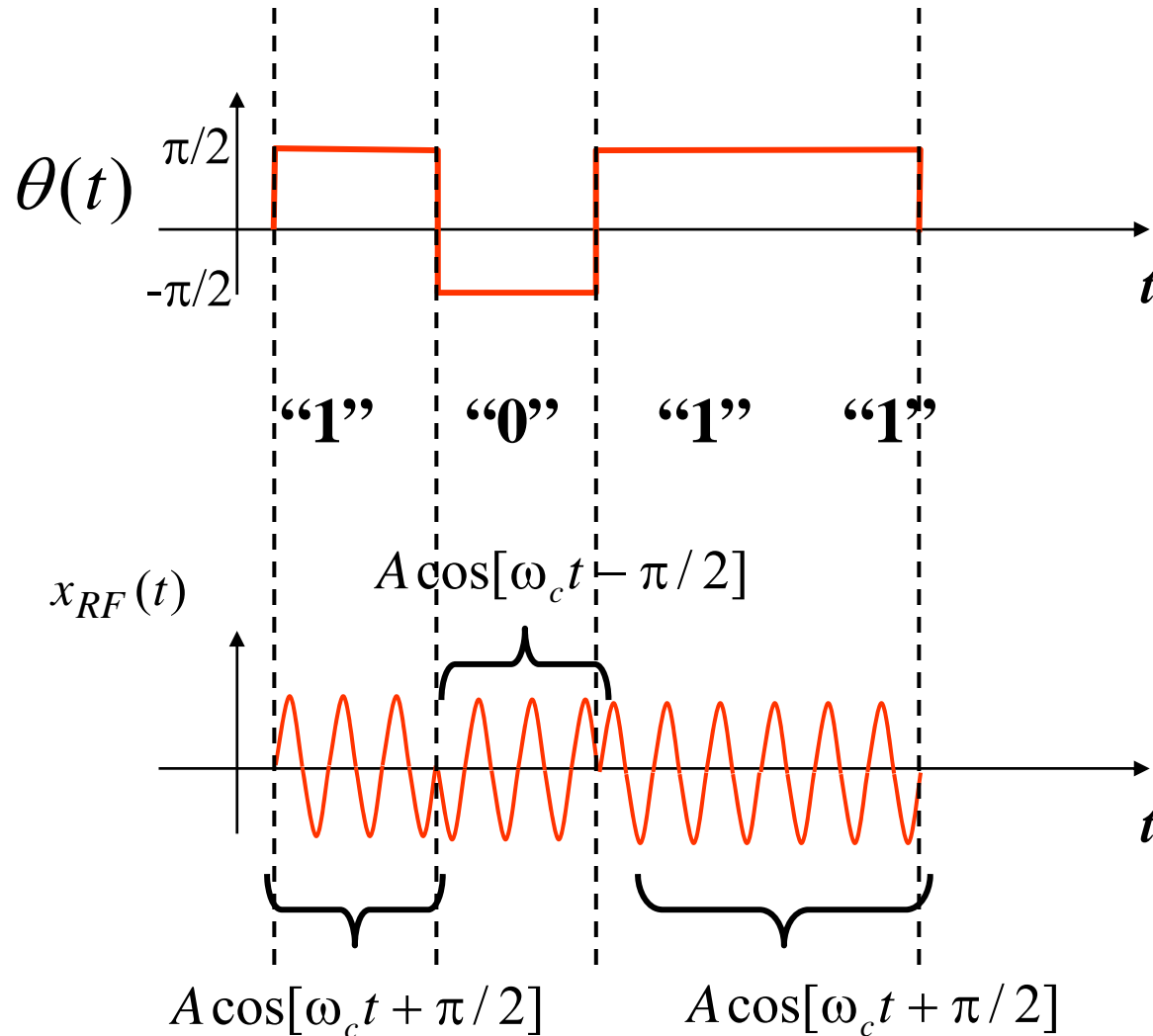\# of Real Values for ELPS   \# of Real Values for BPS

Examples of IQ Signals

Binary Phase Shift Keying (BPSK)
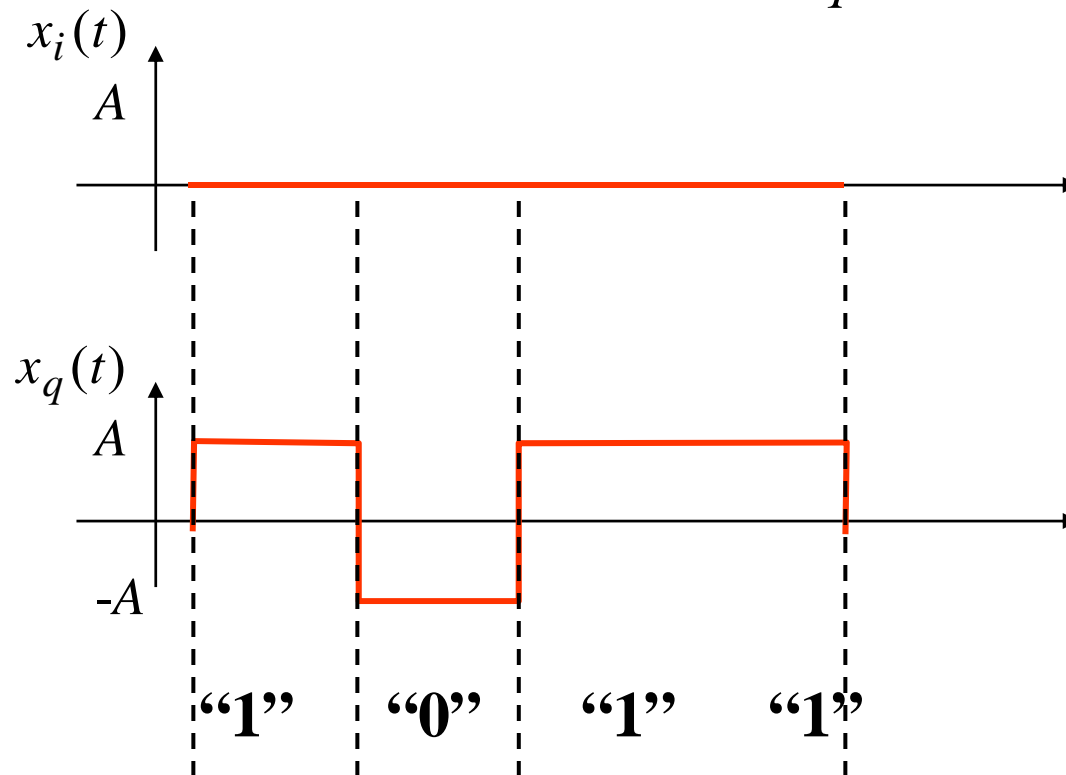
$$x_{RF}(t) = A\cos(\omega_c t + \theta(t))$$

$\theta(t)$

$\pi/2$

$-\pi/2$

t

"1"   "0"   "1"   "1"

$x_{RF}(t)$

$A\cos[\omega_c t - \pi/2]$

t

$A\cos[\omega_c t + \pi/2]$ $A\cos[\omega_c t + \pi/2]$

Binary Phase Shift Keying (Cont.)

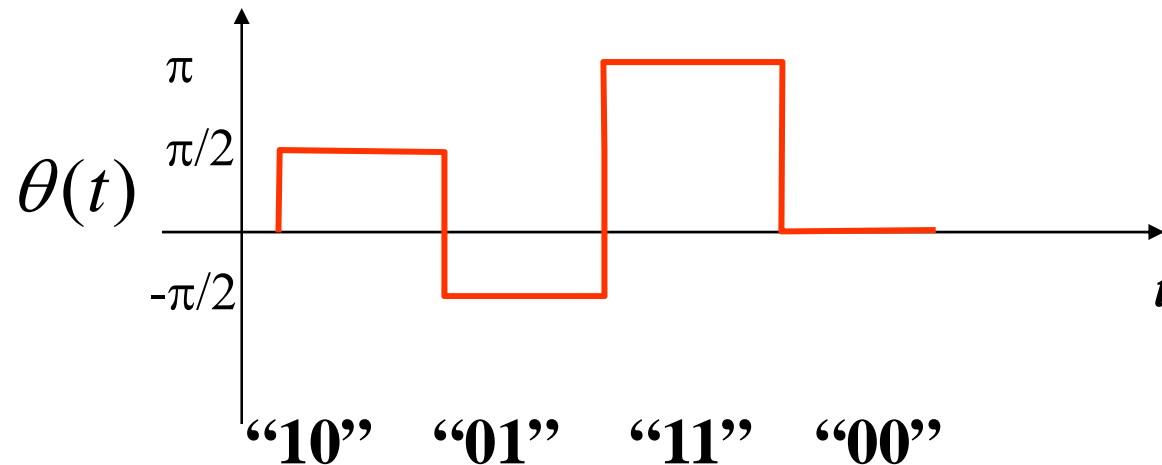What is the LPE signal for BPSK?
Applying the General Result to BPSK gives:

$$x_l(t) = x_i(t) + jx_q(t)$$

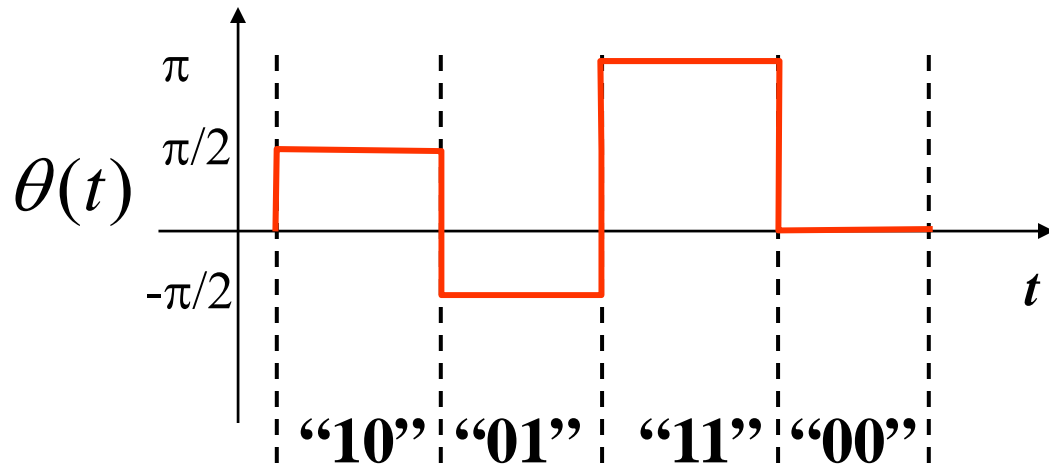$$x_i(t) = A\cos[\theta(t)]$$

$$x_q(t) = A\sin[\theta(t)]$$

Quad Phase Shift Keying (QPSK)
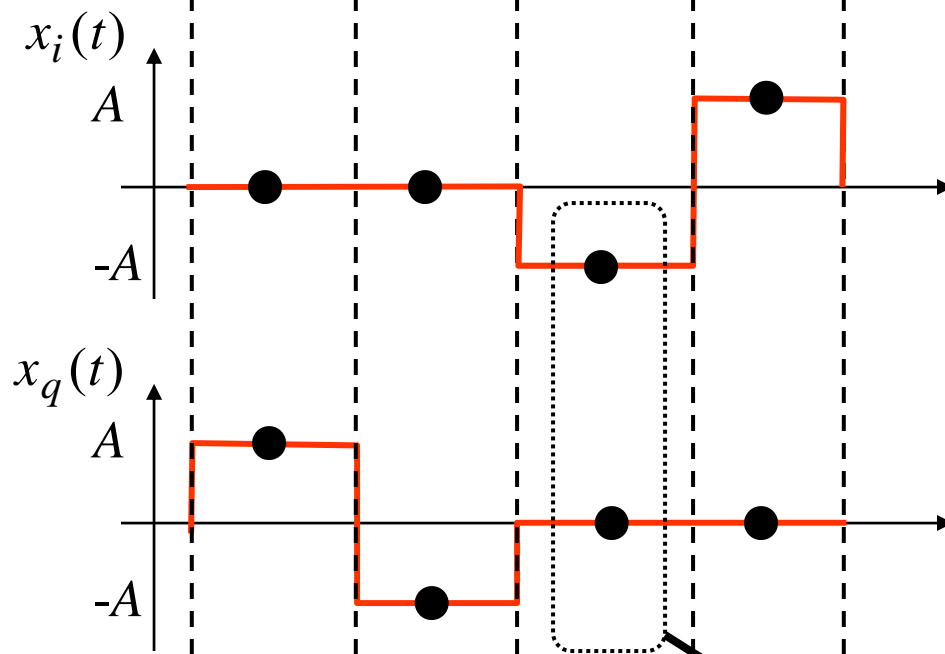
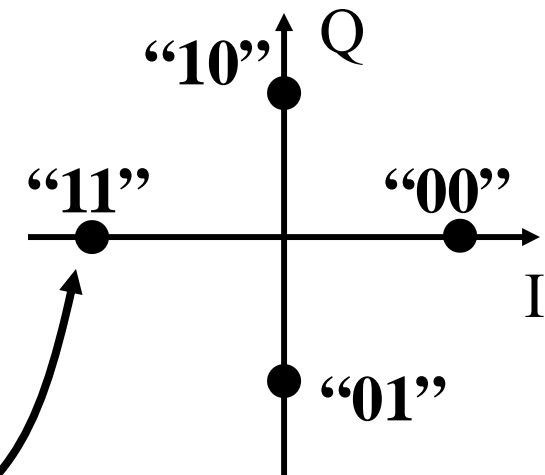$$x_{RF}(t) = A\cos(\omega_c t + \theta(t))$$

Quad Phase Shift Keying (cont.)

$$x_i(t) = A\cos[\theta(t)]$$

$$x_q(t) = A\sin[\theta(t)]$$

$\theta(t)$

π

$\pi/2$

$-\pi/2$

t

"10" "01" "11" "00"

$x_i(t)$

A

$-A$

$x_q(t)$

A

$-A$

"Signal Constellation"

"10" Q

"11" "00"

I

"01"

Radar Pulse Train - Linear FM

Most radars use signals that are trains of pulses.
Those pulses have a carrier frequency that also has frequency modulation imparted on it – and often the frequency is varied linearly with time (during the pulse).

$$x(t) = A(t)\cos[\omega_c t + \theta(t)]$$

$$\omega_i(t) = \frac{d}{dt}[\omega_c t + \theta(t)]$$

$$= \omega_c + \frac{d}{dt}\theta(t)$$

Sinusoid with:
- **Carrier Frequency ω_c**
- **Time-Varying Phase $\theta(t)$**

Gives Inst. Frequency of:
- **Center ω_c**
- **Frequency Deviation $\frac{d}{dt}\theta(t)$**
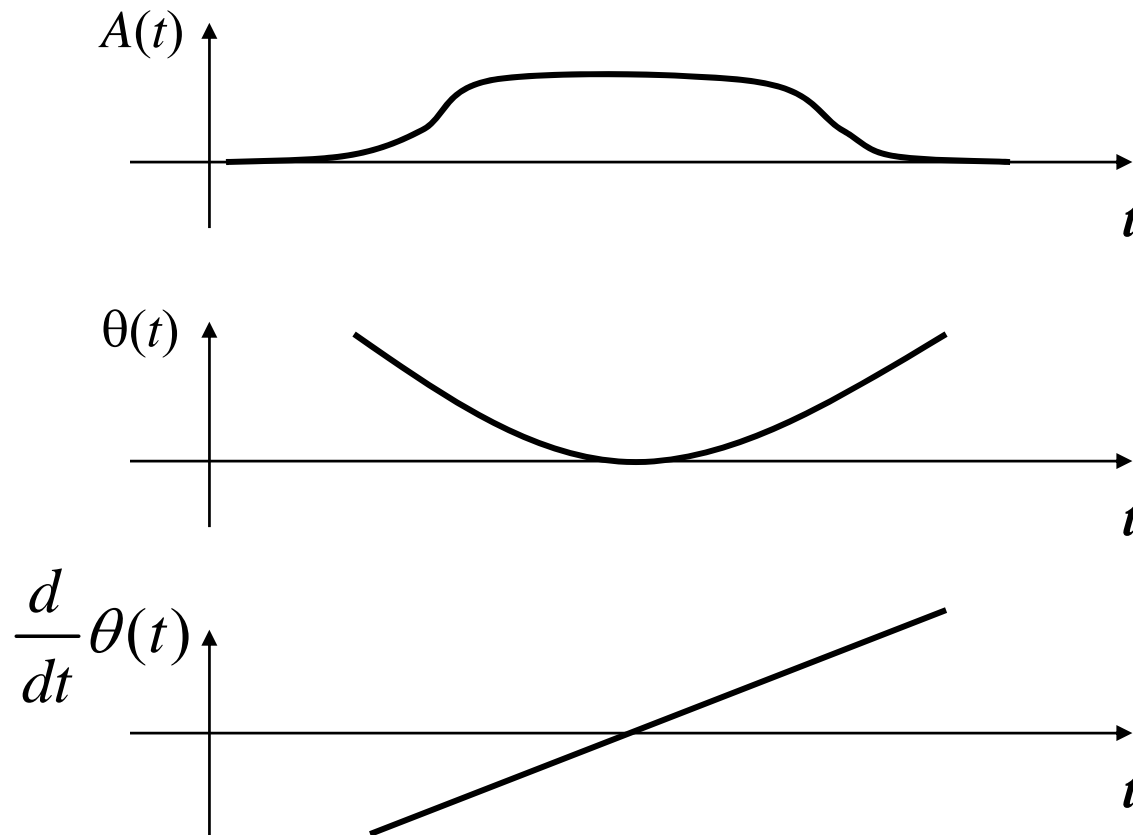
Want this to vary <u>linearly</u>

Radar Pulse Train - Linear FM (cont.)

To get a linear variation of the frequency we need a quadratic variation of the phase:
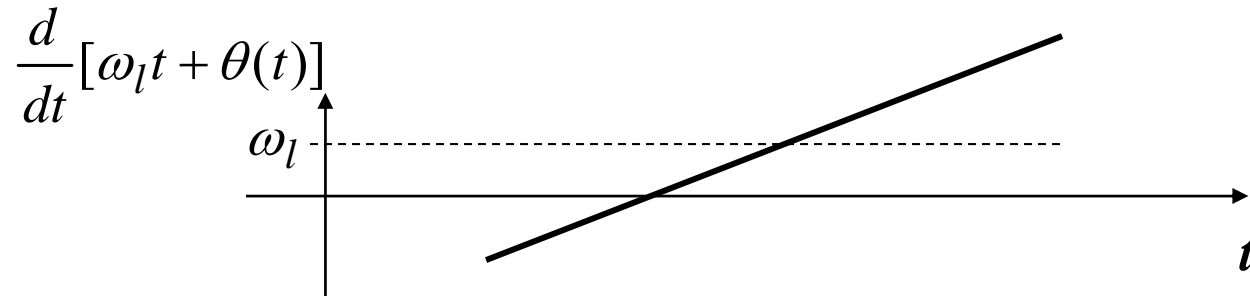
$$x(t) = A(t)\cos[\omega_c t + \theta(t)]$$

Radar Pulse Train - Linear FM (cont.)
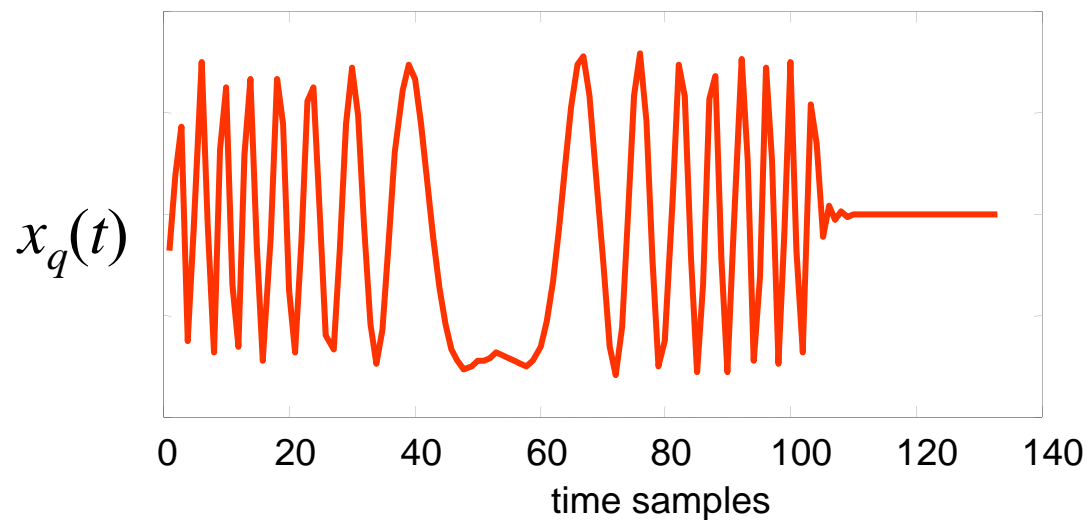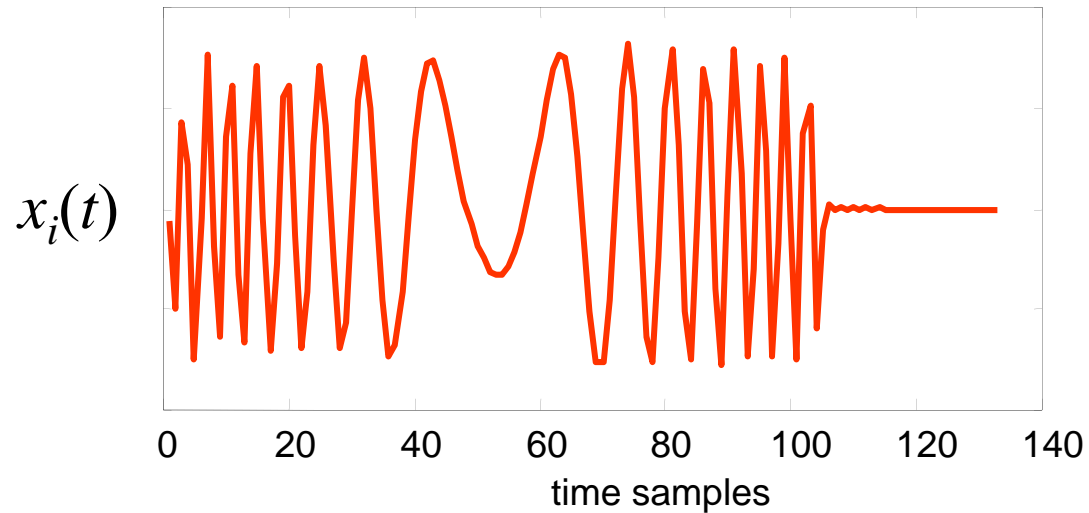
The LPE Signal for the radar signal is:

$$x_l(t) = A(t)e^{j[\omega_l t + \theta(t)]}$$

$$= \underbrace{A(t)\cos[\omega_l t + \theta(t)]}_{x_i(t)} + j\underbrace{A(t)\sin[\omega_l t + \theta(t)]}_{x_q(t)}$$

Small Value: Due to Uncertain Knowledge of ω_c

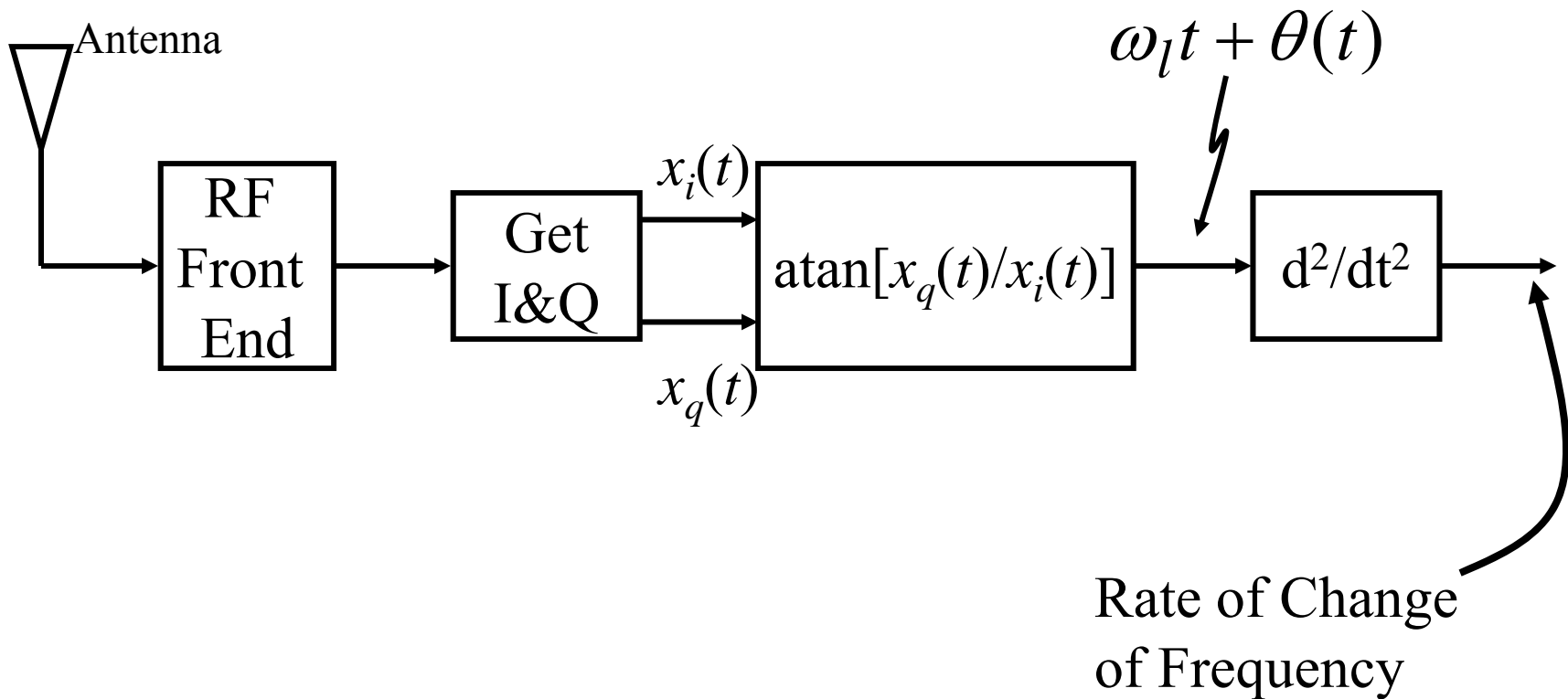$$\frac{d}{dt}[\omega_l t + \theta(t)]$$

ω_l

t

Radar Pulse Train - Linear FM (cont.)

The IQ parts of Linear FM Radar Pulse:

Radar Pulse Train - Linear FM (cont.)

Say you intercepted a radar signal and want to measure its modulation rate (i.e rate of frequency change). You could do it like this:
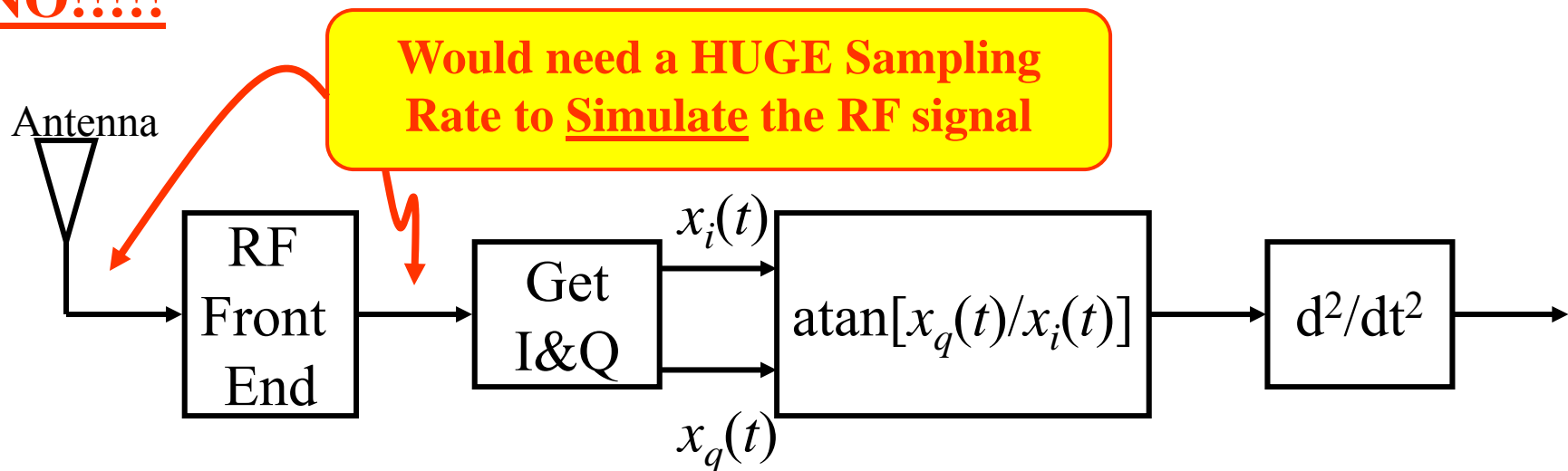
Antenna

$\omega_l t + \theta(t)$

| RF Front End | | Get I&Q | | $\mathrm{atan}[x_q(t)/x_i(t)]$ | | d^2/dt^2 |

$x_i(t)$

$x_q(t)$

Rate of Change of Frequency

Radar Pulse Train - Linear FM (cont.)

Say you wanted to simulate this freq-rate-measurement processing to test how well it works…. How would you write code???

Would you need to simulate the RF signal and the RF Front-End?

NO!!!!!

Would need a HUGE Sampling Rate to Simulate the RF signal

Antenna

RF Front End → Get I&Q → $x_i(t)$ / $x_q(t)$ → atan$[x_q(t)/x_i(t)]$ → d^2/dt^2 →

Simulate using this:

Generate I&Q → $x_i[n]$ / $x_q[n]$ → atan$\{x_q[n]/x_i[n]\}$ → d^2/dt^2 →