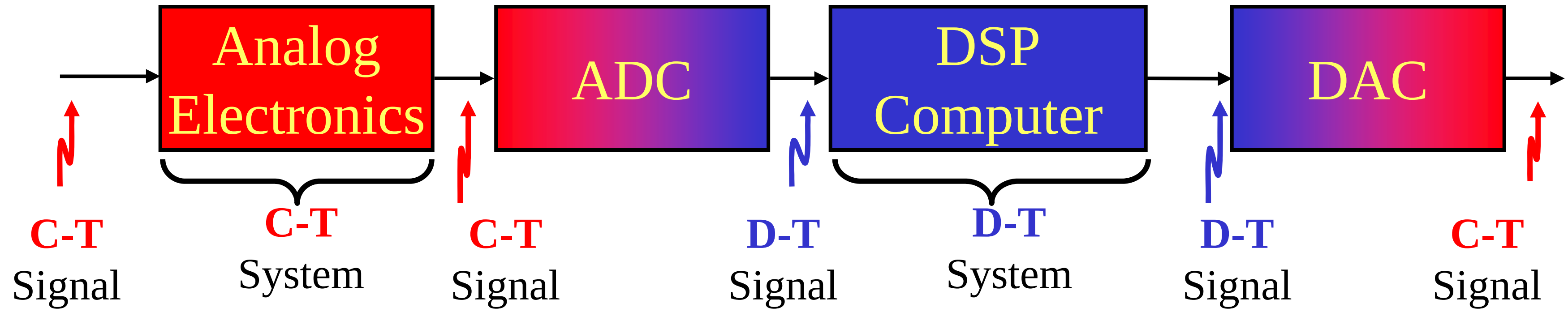
BINGHAMTON UNIVERSITY
State University of New York

EEO 401
Digital Signal Processing
Prof. Mark Fowler

Note Set #13

- Basic Sampling Theory
- Reading Assignment: Sect. 6.1 of Proakis & Manolakis

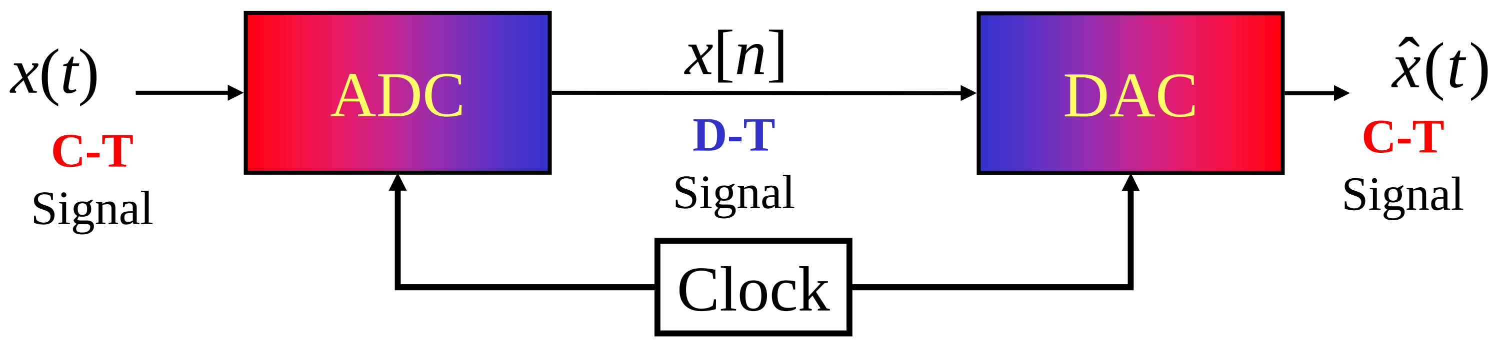
Sampling is Key to Much of Today's Technology



Cell Phones, CDs, Digital Music, Radar, Medical Imaging, etc.!

The first step to see that this is possible:

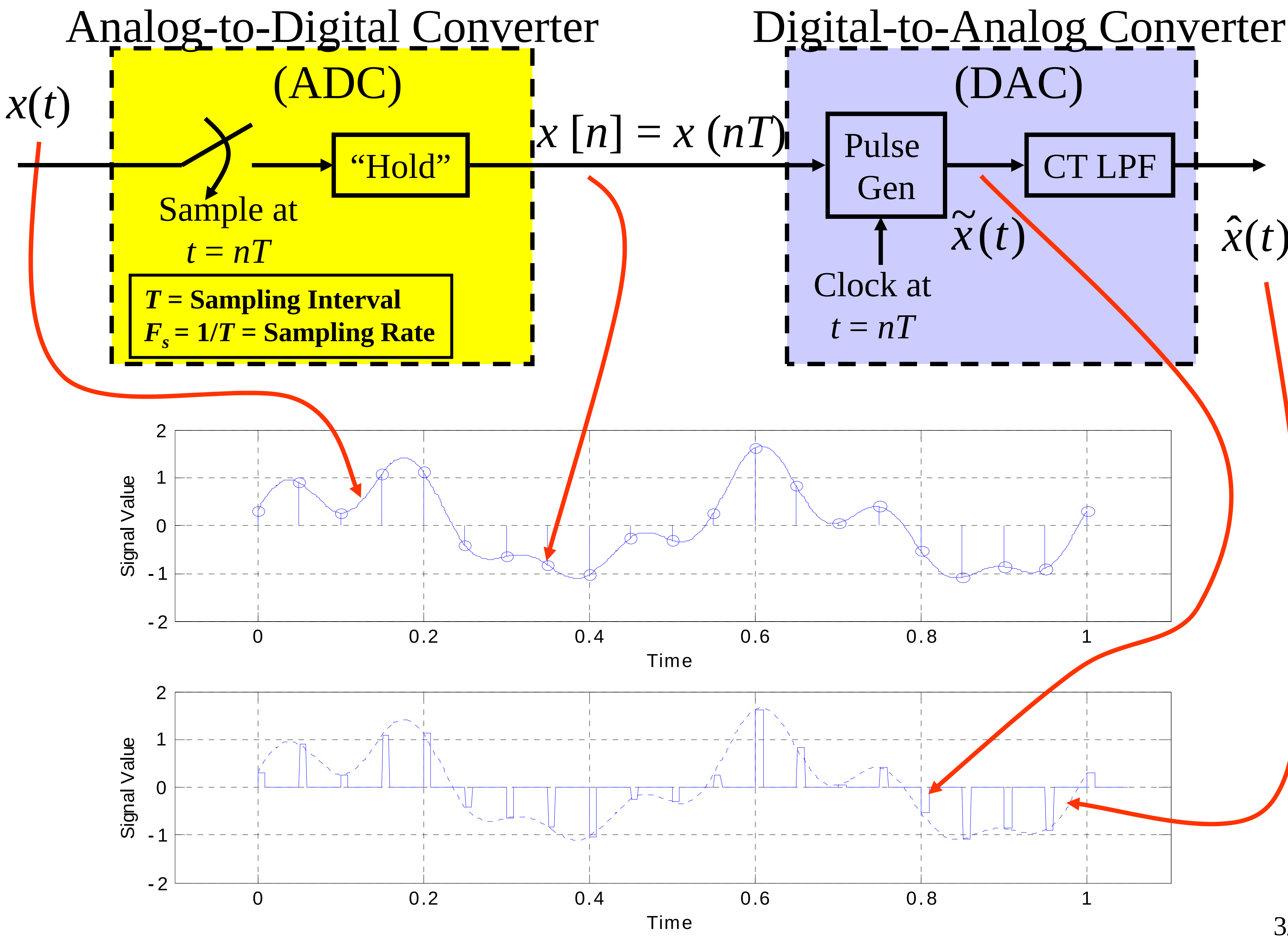
Can we recover the signal from its samples???!!!



Can we make: $\hat{x}(t) = x(t)$**?**

If we *can*… *then* we can process the samples $x[n]$ instead of $x(t)$!!!

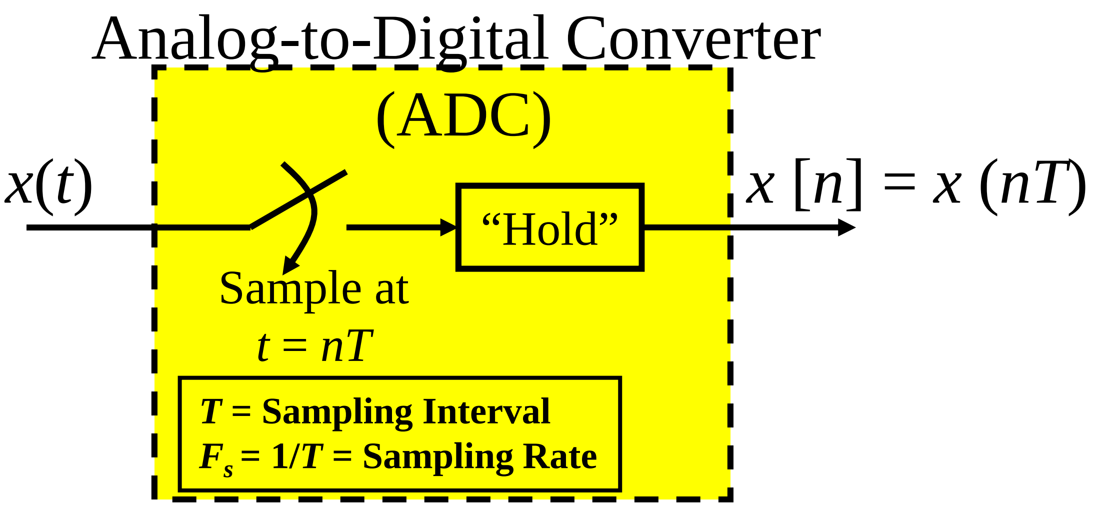
Practical Sampling-Reconstruction Set-Up



Math Model for Sampling (ADC)

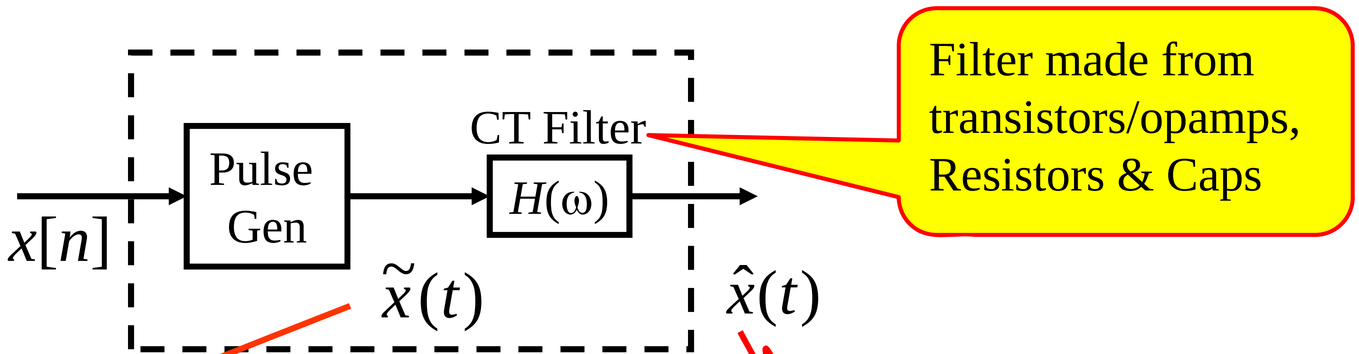
- You learn the circuits in an electronics class
- Here we focus on the "why," so we need math models
- Math Modeling the ADC is **easy**….
  - $x[n] = x(nT)$, so the n^{th} sample is the value of $x(t)$ at $t = nT$

$$\boxed{x[n] = x(t)\Big|_{t=nT} = x(nT)}$$



Math Model for Reconstruction (DAC)

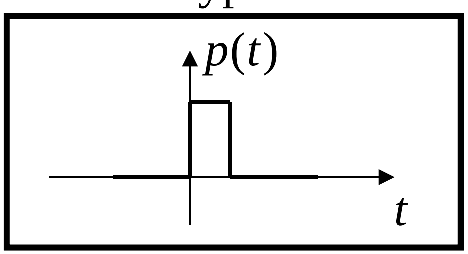
- Math Model for the DAC consists of two parts:
  - converting a DT sequence (of numbers) into a CT pulse train
  - “smoothing” out the pulse train using a lowpass filter

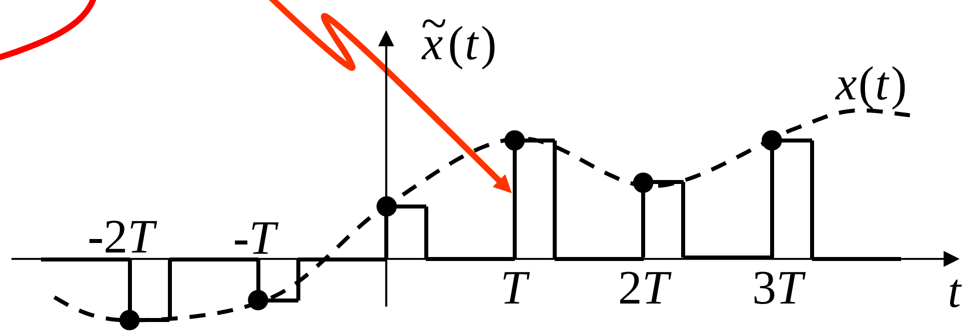


$$\tilde{x}(t) = \sum_{n=-\infty}^{\infty} x(nT)\underbrace{p(t-nT)}$$

$$\hat{X}(\omega) = \tilde{X}(\omega)H(\omega)$$

“Prototype” Pulse





<u>"Impulse Sampling" Model for DAC</u>

Now we have a good model that handles quite well what REALLY happens inside a DAC… but we simplify it !!!!

$$\boxed{\text{To Ease Analysis: Use } \quad p(t) = \delta(t)}$$

Why???? 1. Because delta functions are <u>EASY</u> to analyze!!!

2. Because it leads to the best possible results (see later!)

3. We can easily account for real-life pulses later!!

$$\boxed{p(t) = \delta(t)} \Rightarrow \boxed{\tilde{x}(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)}$$

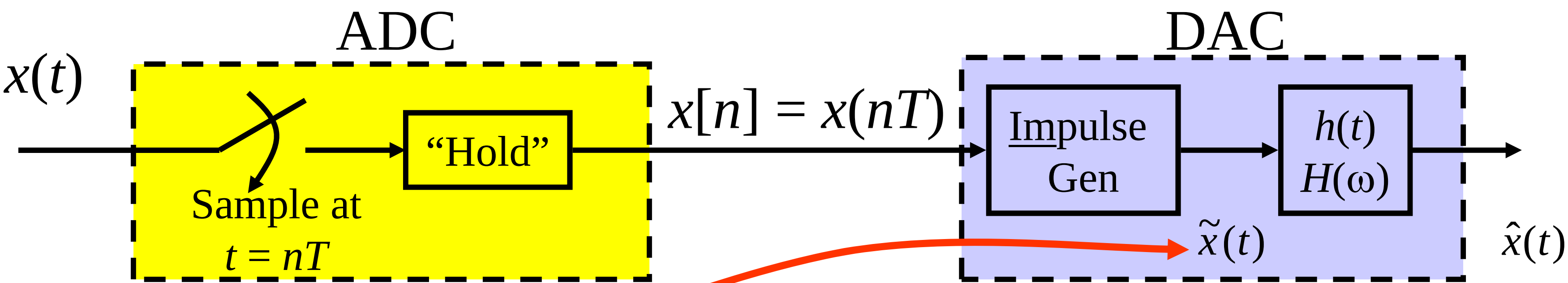
In this form… this is called the "<u>Impulse</u> Sampled" signal.

Now.. Using property of delta function we can also write…

$$\boxed{\tilde{x}(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT)}$$

Sampling Analysis (p. 1)

Analysis will be done using the Impulse Sampling Math Model

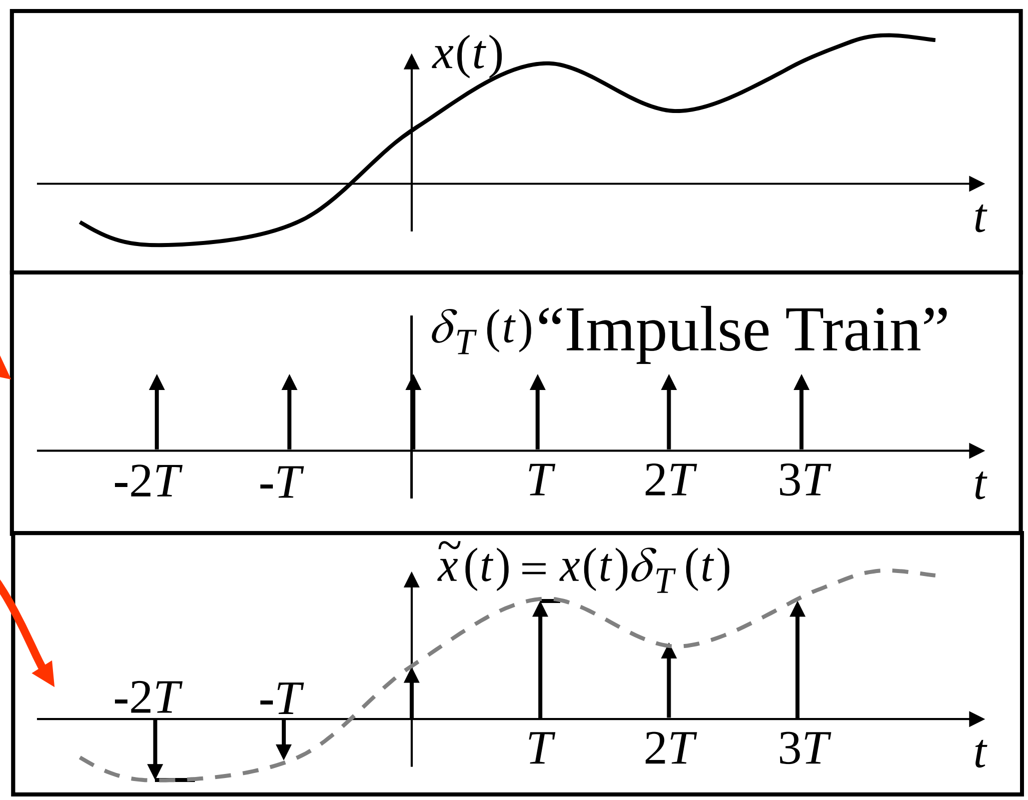


$$\tilde{x}(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$= x(t)\delta_T(t)$$

Impulse Sampled Signal

Note: we are using the “impulse sampling” model in the DAC not the ADC!!!



Sampling Analysis (p. 2)

Goal = Determine Under What Conditions We Get:

Reconstructed CT Signal = *Original* CT Signal

$$\hat{x}(t) = x(t)$$

Approach: 1. Find the FT of the signal $\tilde{x}(t)$

2. Use Freq. Response of Filter to get $\hat{X}(\omega) = \tilde{X}(\omega)H(\omega)$

3. Look to see what is needed to make $\hat{X}(\omega) = X(\omega)$

Sampling Analysis (p. 3)

Step #1: Hmmm… well $\delta_T(t)$ is periodic with period T so we COULD expand it as a Fourier series:

$$\delta_T(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk2\pi F_s t}$$

Period = T sec
Fund. Freq = F_s = 1/T Hz

So… what are the FS coefficients???

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta_T(t) e^{-jk2\pi F_s t} = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jk2\pi F_s t}$$

Only one delta inside a single period

$$= \frac{1}{T}\left[e^{-jk2\pi F_s t}\right]_{t=0} = \frac{1}{T}$$

By sifting property of the delta function!!!

So… an alternate model for $\delta_T(t)$ is

$$\delta_T(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} e^{jk2\pi F_s t}$$

Sampling Analysis (p. 4)

So we now have….

$$\tilde{x}(t) = x(t)\delta_T(t)$$

$$= x(t)\left[\frac{1}{T}\sum_{k=-\infty}^{\infty} e^{jk2\pi F_s t}\right]$$

Use FS Result

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} x(t)e^{jk2\pi F_s t}$$

By frequency shift property of FT… each term is a frequency shifted version of the original signal!!!

So using the frequency shift property of the FT gives:

$$\boxed{\tilde{X}(f) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(f + kF_s)}$$

Extremely Important Result… the basis of all understanding of sampling!!!

$$\tilde{X}(f) = \frac{1}{T}\left[\cdots + X(f-2F_s) + X(f-F_s) + \boxed{X(f)} + X(f+F_s) + X(f+2F_s) + \cdots\right]$$

Original FT

Shifted Replicas

<u>Sampling Analysis (p. 5)</u>

So… the **<u>BIG Thing</u>** we've just found out is that:

the impulse sampled signal (inside the DAC) has a FT that consists of the <u>original signal's FT and frequency-shifted version of it</u> (where the frequency shifts are by integer multiples of the sampling rate F_s):

$$\tilde{X}(f) = \frac{1}{T}\left[\cdots + X(f - 2F_s) + X(f - F_s) + \boxed{X(f)} + X(f + F_s) + X(f + 2F_s) + \cdots\right]$$

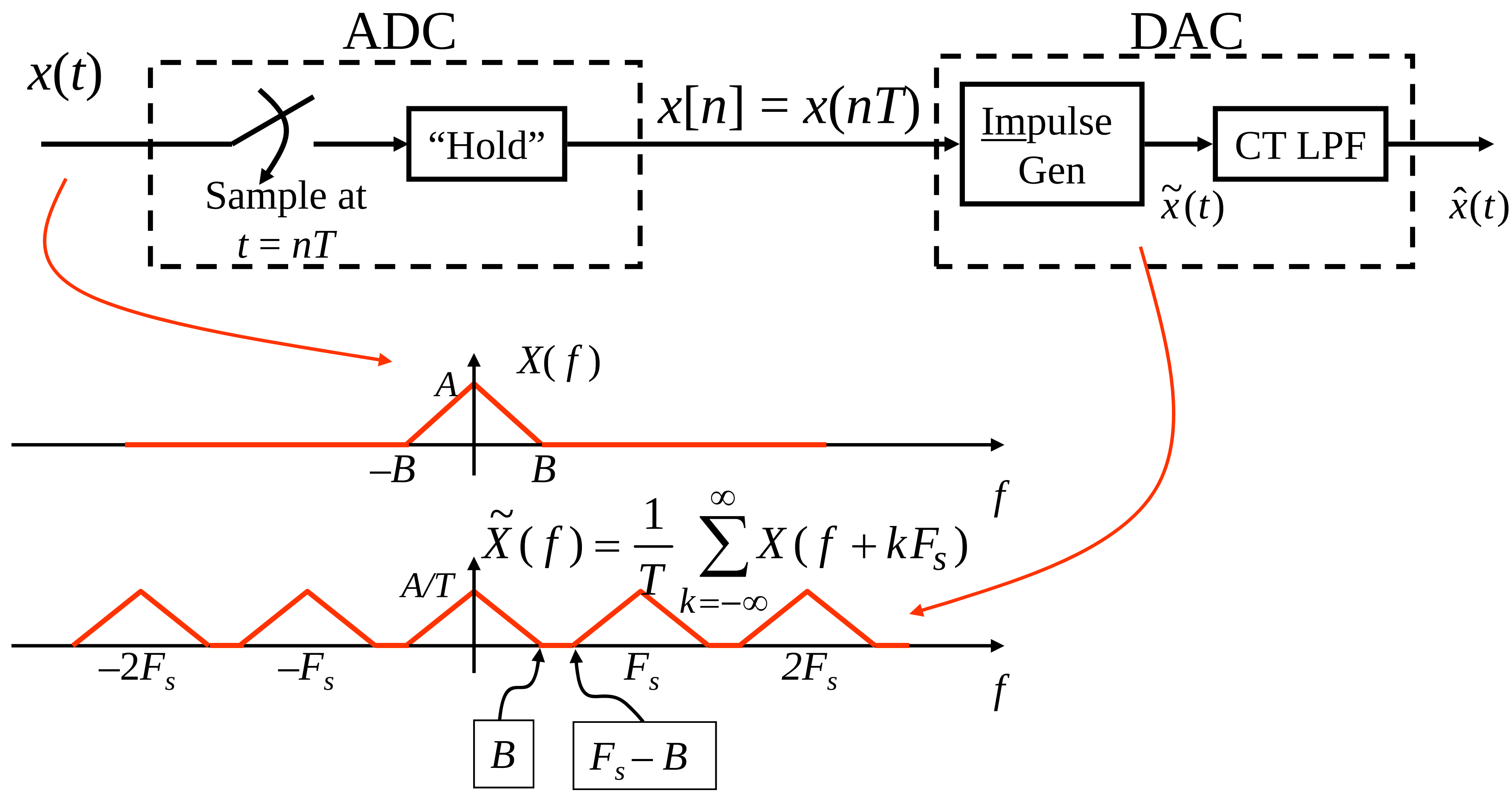
Original FT

This result allows us to see how to make sampling work …

By "work" we mean: how to ensure that even though we only have samples of the signal, we can still get perfect reconstruction of the original signal…. at least in theory!!

The figure on the next page shows how….

Sampling Analysis (p. 6)

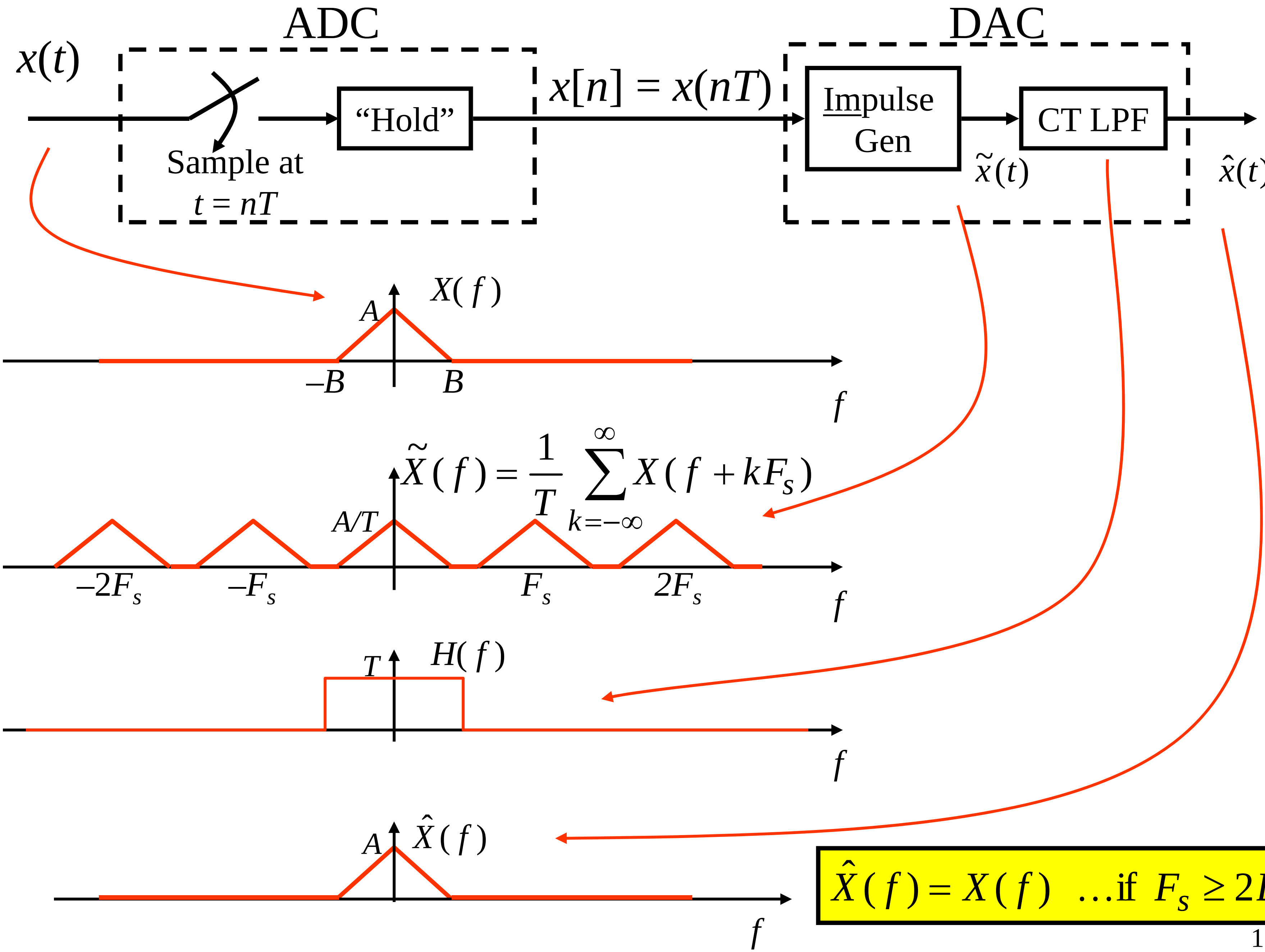


To ensure that the replicas don't overlap the original….

we need $F_s - B \geq B$ or equivalently… $\boxed{F_s \geq 2B}$

When there is no overlap, the original spectrum is left "unharmed" and can be recovered using a CT LPF (as seen on the next page).

Sampling Analysis (p. 7)



$$\hat{X}(f) = X(f) \quad \ldots\text{if } F_s \geq 2B$$

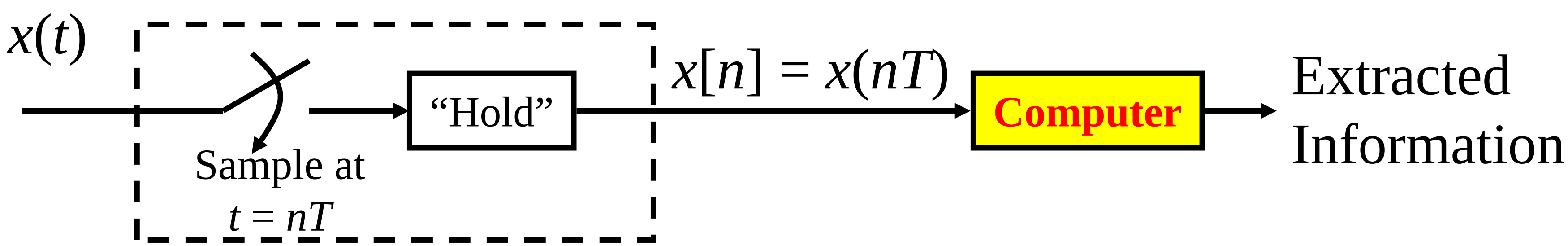
Sampling Analysis Result

What this analysis says:

Sampling Theorem: A **bandlimited** signal with BW = B Hz is completely defined by its samples as long as they are taken at a rate $F_s \geq 2B$ (samples/second).

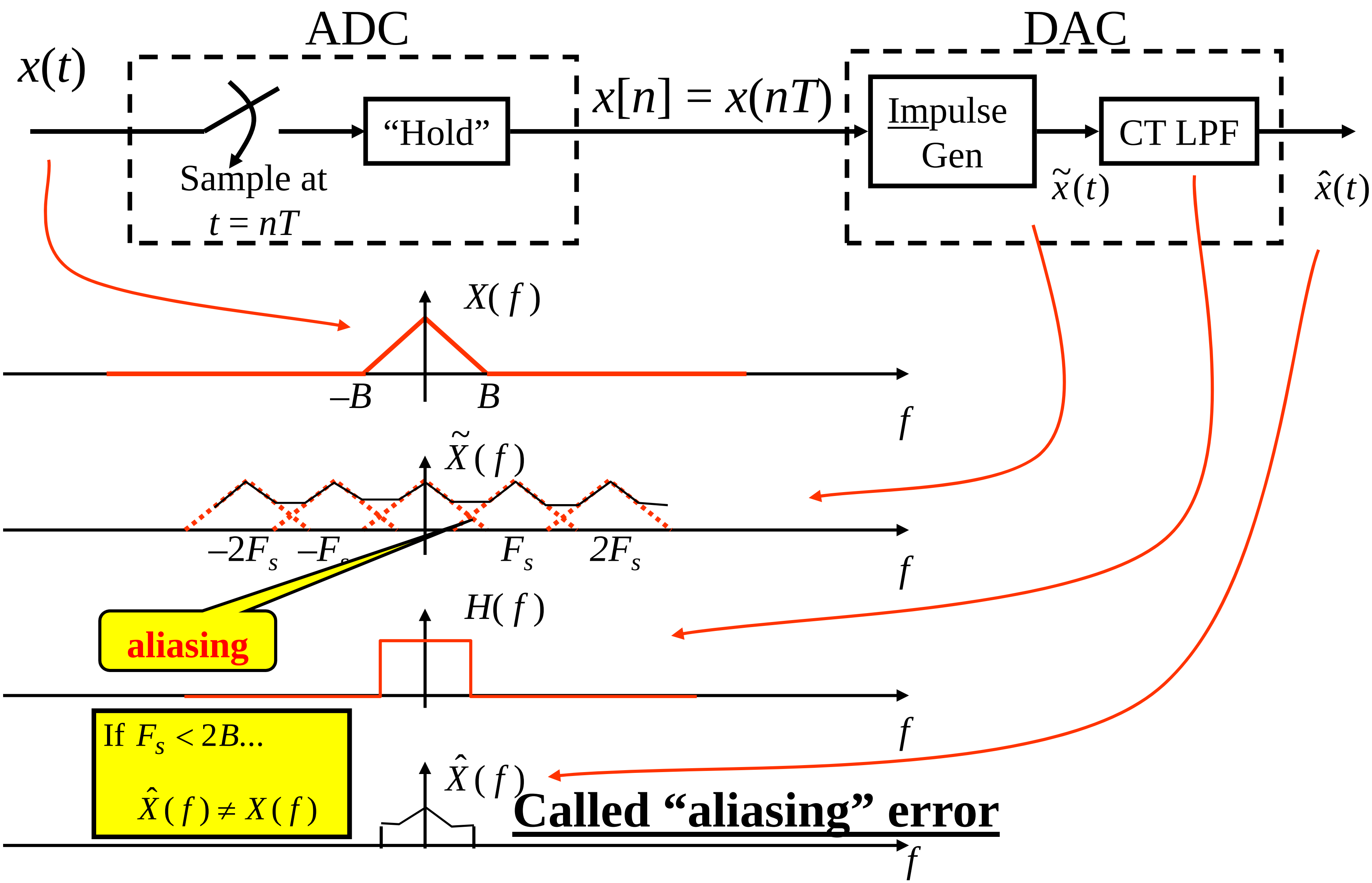
Impact: To extract the info from a **bandlimited** signal we only need to operate on its (properly taken) samples

➔ **Then can use a computer to process signals!!!**



This math result (published in the late 1940s!) is the foundation of:
…CD's, MP3's, digital cell phones, etc….

"Aliasing" Analysis: What if samples are *not* taken fast enough???



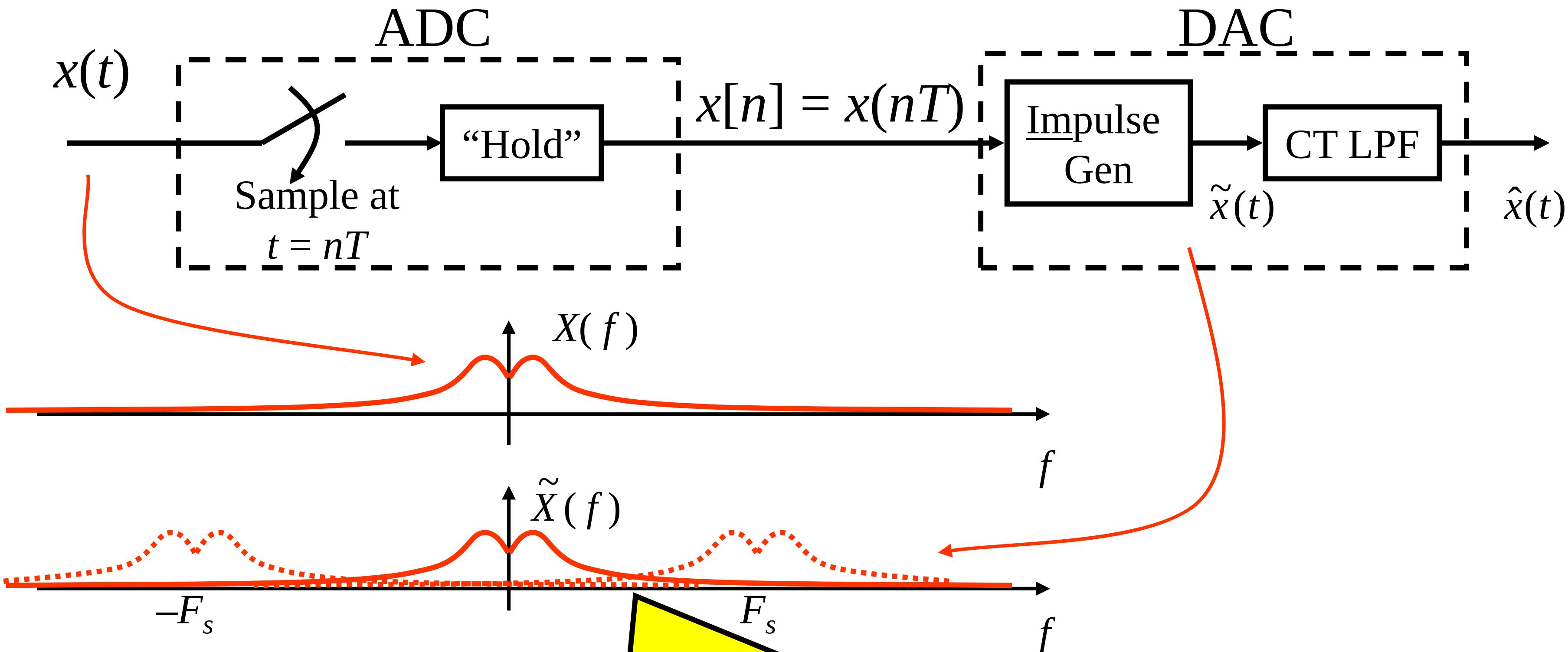
If $F_s < 2B$...

$\hat{X}(f) \neq X(f)$

Called "aliasing" error

To enable error-free reconstruction, a signal bandlimited to B Hz <u>must</u> be sampled faster than 2B samples/sec

"Aliasing" Analysis: What if the signal is NOT BANDLIMITED???



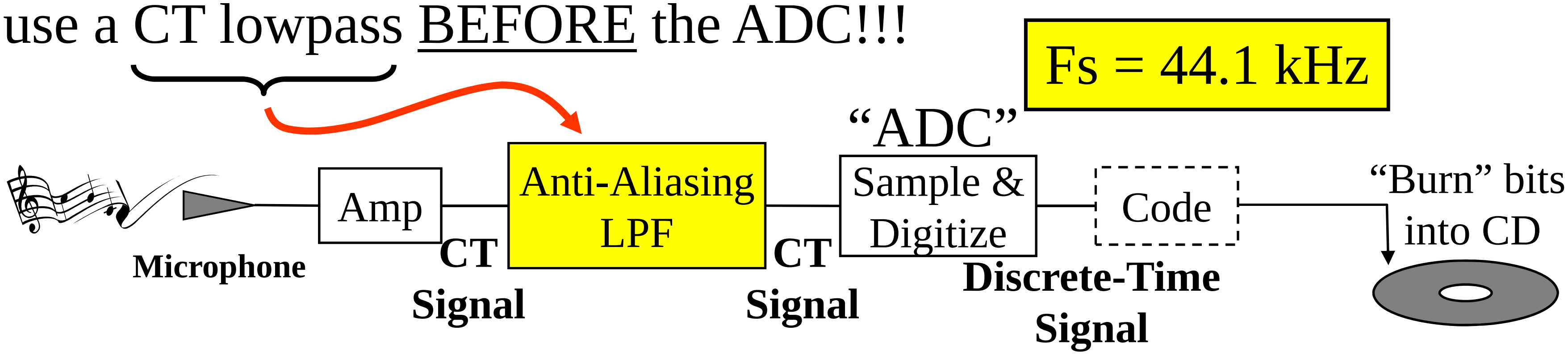
<u>For Non-BL Signal</u> Aliasing always happens regardless of F_s value

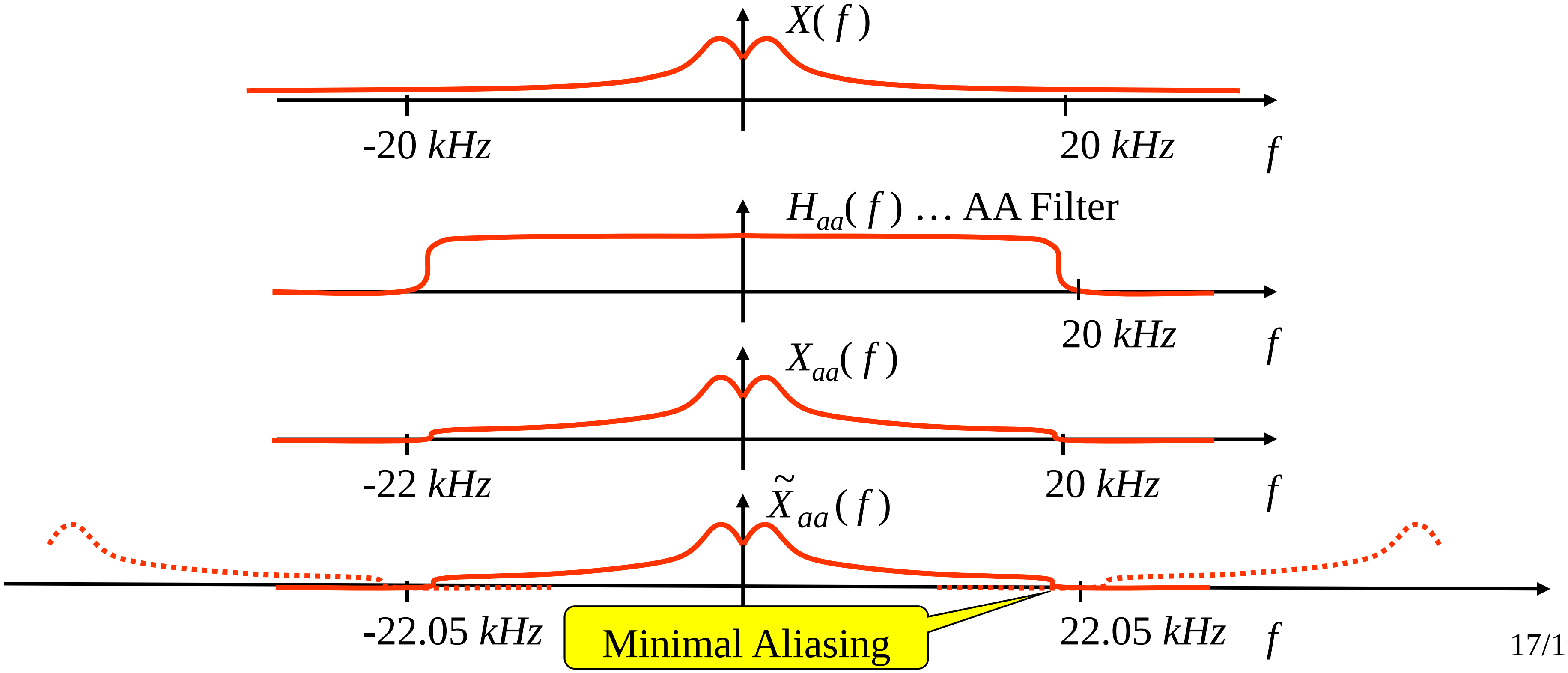
<u>All</u> practical signal are <u>Non</u>-BL!!!!

… so we choose F_s to minimize the aliasing to an level acceptable for the specific application

Practical Sampling: Use of Anti-Aliasing Filter

In practice it is important to avoid excessive aliasing. So we use a CT lowpass **BEFORE** the ADC!!!





Some Sampling Terminology

F_s is called the sampling rate. Its unit is samples/sec which is often "equivalently" expressed as Hz.

The minimum sampling rate of F_s = 2B samples/sec is called the Nyquist Rate.

Sampling at the Nyquist rate is called Critical Sampling.

Sampling faster than the Nyquist rate is called Over Sampling

Sampling slower than the Nyquist rate is called Under Sampling

Note: Critical sampling is only possible if an IDEAL lowpass filter is used…. so in practice we generally need to choose a sampling rate somewhat above the Nyquist rate (e.g., $2.2B$); the choice depends on the application.

Summary of Sampling

- Math Model for Impulse Sampling (inside the DAC) says
  - The FT of the impulse sampled signal has spectral replicas spaced F_s Hz apart
  - This math result drives all of the insight into practical aspects
- Theory says for a BL'd Signal with BW = B Hz
  - It is completely defined by samples taken at a rate $F_s \geq 2B$
  - Then… Perfect reconstruction can be achieved using an ideal LPF reconstruction filter (i.e., the filter inside the DAC)
- Theory says for a Practical Signal…
  - Practical signals aren't bandlimited… so use an Anti-Aliasing lowpass filter BEFORE the ADC
  - Because the A-A LPF is not ideal there will still be some aliasing
    - Design the A-A LPF to give acceptably low aliasing error for the expected types of signals