Note Set #13

- Basic Sampling Theory
- Reading Assignment: Sect. 6.1 of Proakis & Manolakis
Sampling is Key to Much of Today’s Technology

Cell Phones, CDs, Digital Music, Radar, Medical Imaging, etc.!

The first step to see that this is possible:

**Can we recover the signal from its samples??!!**

Can we make: \( \hat{x}(t) = x(t) \)??

If we *can*... *then* we can process the samples \( x[n] \) instead of \( x(t) \)!!!
Practical Sampling-Reconstruction Set-Up

Analog-to-Digital Converter (ADC)

\[ x(t) \rightarrow \text{"Hold"} \rightarrow x[n] = x(nT) \]

Sample at \( t = nT \)

\( T = \) Sampling Interval
\( F_s = 1/T = \) Sampling Rate

Digital-to-Analog Converter (DAC)

\( \tilde{x}(t) \rightarrow \text{Pulse Gen} \rightarrow \text{CT LPF} \rightarrow \hat{x}(t) \)

Clock at \( t = nT \)
Math Model for Sampling (ADC)

• You learn the circuits in an electronics class
• Here we focus on the “why,” so we need math models
• Math Modeling the ADC is easy….
  – $x[n] = x(nT)\,$, so the $n$th sample is the value of $x(t)$ at $t = nT$

$$x[n] = x(t)\bigg|_{t=nT} = x(nT)$$

Analog-to-Digital Converter (ADC)

- Sample at $t = nT$
- "Hold"

$T = \text{Sampling Interval}$
$F_s = 1/T = \text{Sampling Rate}$
Math Model for Reconstruction (DAC)

- Math Model for the DAC consists of two parts:
  - converting a DT sequence (of numbers) into a CT pulse train
  - “smoothing” out the pulse train using a lowpass filter

\[
\tilde{x}(t) = \sum_{n=-\infty}^{\infty} x(nT) p(t - nT)
\]

\[
\hat{X}(\omega) = \tilde{X}(\omega) H(\omega)
\]

“Prototype” Pulse

Filter made from transistors/opamps, Resistors & Caps
“Impulse Sampling” Model for DAC

Now we have a good model that handles quite well what REALLY happens inside a DAC… but we simplify it !!!!

To Ease Analysis: Use $p(t) = \delta(t)$

Why????
1. Because delta functions are EASY to analyze!!!
2. Because it leads to the best possible results (see later!)
3. We can easily account for real-life pulses later!!

To Ease Analysis: Use $p(t) = \delta(t)$

\[
\tilde{x}(t) = \sum_{n=-\infty}^{\infty} x(nT) \delta(t - nT)
\]

In this form… this is called the ‘Impulse Sampled” signal.

Now.. Using property of delta function we can also write…

\[
\tilde{x}(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT)
\]
Sampling Analysis (p. 1)

Analysis will be done using the Impulse Sampling Math Model

\[ x(t) \xrightarrow{ADC} \text{Sample at } t = nT \]

\[ x[n] = x(nT) \xrightarrow{DAC} \tilde{x}(t) \]

\[ \tilde{x}(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) \]

\[ = x(t)\delta_T(t) \]

Impulse Sampled Signal

\[ \text{Note: we are using the “impulse sampling” model in the DAC not the ADC!!!} \]
**Sampling Analysis (p. 2)**

**Goal** = Determine Under What Conditions We Get:

*Reconstructed CT Signal = Original CT Signal*

\[ \hat{x}(t) = x(t) \]

**Approach:**
1. Find the FT of the signal \( \tilde{x}(t) \)
2. Use Freq. Response of Filter to get \( \hat{X}(\omega) = \tilde{X}(\omega)H(\omega) \)
3. Look to see what is needed to make \( \hat{X}(\omega) = X(\omega) \)
Sampling Analysis (p. 3)

Step #1: Hmmm… well $\delta_T(t)$ is periodic with period $T$ so we COULD expand it as a Fourier series:

$$\delta_T(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk2\pi F_s t}$$

So… what are the FS coefficients???

$$c_k = \frac{1}{T} \int_{-T/2}^{T/2} \delta_T(t)e^{-jk2\pi F_s t} = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t)e^{-jk2\pi F_s t}$$

$$= \frac{1}{T} \left[ e^{-jk2\pi F_s t} \right]_{t=0} = \frac{1}{T}$$

So… an alternate model for $\delta_T(t)$ is

$$\delta_T(t) = \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{jk2\pi F_s t}$$
Sampling Analysis (p. 4)

So we now have….

\[
\tilde{x}(t) = x(t)\delta_T(t)
\]

\[
= x(t) \left[ \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{j2\pi F_s t_k} \right]
\]

\[
= \frac{1}{T} \sum_{k=-\infty}^{\infty} x(t)e^{j2\pi F_s t_k}
\]

So using the frequency shift property of the FT gives:

\[
\tilde{X}(f) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(f + kF_s)
\]

\[
\tilde{X}(f) = \frac{1}{T}[\cdots + X(f - 2F_s) + X(f - F_s) + \boxed{X(f)} + X(f + F_s) + X(f + 2F_s) + \cdots]
\]

Use FS Result

By frequency shift property of FT… each term is a frequency shifted version of the original signal!!!

Extremely Important Result… the basis of all understanding of sampling!!!

Original FT

Shifted Replicas
Sampling Analysis (p. 5)

So… the **BIG Thing** we’ve just found out is that:

*the impulse sampled signal (inside the DAC) has a FT that consists of the original signal’s FT and frequency-shifted version of it (where the frequency shifts are by integer multiples of the sampling rate $F_s$):*

$$\tilde{X}(f) = \frac{1}{T} \left[ \cdots + X(f - 2F_s) + X(f - F_s) + X(f) + X(f + F_s) + X(f + 2F_s) + \cdots \right]$$

This result allows us to see how to make sampling work …

By “work” we mean: how to ensure that even though we only have samples of the signal, we can still get perfect reconstruction of the original signal…. at least in theory!!

The figure on the next page shows how…. 
To ensure that the replicas don’t overlap the original….
we need $F_s - B \geq B$ or equivalently… $F_s \geq 2B$

When there is no overlap, the original spectrum is left “unharmed”
and can be recovered using a CT LPF (as seen on the next page).
Sampling Analysis (p. 7)

\[ x(t) \xrightarrow{\text{Sample at } t = nT} \text{ADC} \xrightarrow{\text{"Hold"}} x[n] = x(nT) \xrightarrow{\text{Impulse Gen}} \tilde{x}(t) \xrightarrow{\text{CT LPF}} \hat{x}(t) \]

\[ X(f) \quad -B \quad B \]

\[ \tilde{X}(f) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(f + kF_s) \]

\[ H(f) \quad T \quad \hat{X}(f) \quad A \quad \hat{X}(f) = X(f) \quad \text{...if } F_s \geq 2B \]
Sampling Analysis Result

What this analysis says:

**Sampling Theorem:** A **bandlimited** signal with BW = $B$ Hz is completely defined by its samples as long as they are taken at a rate $F_s \geq 2B$ (samples/second).

**Impact:** To extract the info from a **bandlimited** signal we only need to operate on its (properly taken) samples

⇒ Then can use a computer to process signals!!!

\[ x(t) \xrightarrow{\text{Sample at } t = nT} \text{“Hold”} \xrightarrow{x[n] = x(nT)} \text{Computer} \xrightarrow{\text{Extracted Information}} \]

This **math** result (published in the late 1940s!) is the foundation of:

...CD’s, MP3’s, digital cell phones, etc....
“Aliasing” Analysis: What if samples are not taken fast enough???

To enable error-free reconstruction, a signal bandlimited to \( B \) Hz must be sampled faster than \( 2B \) samples/sec.
“Aliasing” Analysis: What if the signal is NOT BANDLIMITED???

For Non-BL Signal Aliasing always happens regardless of $F_s$ value

All practical signal are Non-BL!!!!

… so we choose $F_s$ to minimize the aliasing to an level acceptable for the specific application
Practical Sampling: Use of Anti-Aliasing Filter

In practice it is important to avoid excessive aliasing. So we use a CT lowpass **BEFORE** the ADC!!!

Fs = 44.1 kHz

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**Anti-Aliasing LPF**

Microphone

Amp

CT Signal

Anti-Aliasing Filter

Sample & Digitize

“ADC”

Discrete-Time Signal

Code

“Burn” bits into CD

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**Signal Flow**

Microphone → Amp → Anti-Aliasing Filter → Sample & Digitize → “ADC” → Code → “Burn” bits into CD

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**Frequency Diagrams**

- **$X(f)$**
  - $-20 \text{ kHz}$ to $20 \text{ kHz}$

- **$H_{aa}(f)$ … AA Filter**
  - $-20 \text{ kHz}$ to $20 \text{ kHz}$

- **$X_{aa}(f)$**
  - $-22 \text{ kHz}$ to $20 \text{ kHz}$

- **$\tilde{X}_{aa}(f)$**
  - $-22.05 \text{ kHz}$ to $22.05 \text{ kHz}$

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**Minimal Aliasing**

- $-22.05 \text{ kHz}$ to $22.05 \text{ kHz}$

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Some Sampling Terminology

$F_s$ is called the sampling rate. Its unit is samples/sec which is often “equivocantly” expressed as Hz.

The minimum sampling rate of $F_s = 2B$ samples/sec is called the Nyquist Rate.

Sampling at the Nyquist rate is called Critical Sampling.

Sampling faster than the Nyquist rate is called Over Sampling.

Sampling slower than the Nyquist rate is called Under Sampling.

Note: Critical sampling is only possible if an IDEAL lowpass filter is used…. so in practice we generally need to choose a sampling rate somewhat above the Nyquist rate (e.g., $2.2B$); the choice depends on the application.
Summary of Sampling

• Math Model for Impulse Sampling (inside the DAC) says
  – The FT of the impulse sampled signal has spectral replicas spaced $F_s$ Hz apart
  – This math result drives all of the insight into practical aspects

• Theory says for a BL’d Signal with BW = $B$ Hz
  – It is completely defined by samples taken at a rate $F_s \geq 2B$
  – Then… Perfect reconstruction can be achieved using an ideal LPF reconstruction filter (i.e., the filter inside the DAC)

• Theory says for a Practical Signal…
  – Practical signals aren’t bandlimited… so use an Anti-Aliasing lowpass filter BEFORE the ADC
  – Because the A-A LPF is not ideal there will still be some aliasing
    • Design the A-A LPF to give acceptably low aliasing error for the expected types of signals