BINGHAMTON UNIVERSITY
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EEO 401
Digital Signal Processing

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Note Set #10

- Fourier Analysis for DT Signals
- Reading Assignment: Sect. 4.2 & 4.4 of Proakis & Manolakis

Much of Ch. 4 should be review… so you are expected to read it to refresh your memory. We'll focus on a few topics you might not have seen before

<u>Fourier Series for DT Periodic Signals (DTFS)</u>

The idea here is the same as FS for CT periodic signals… just a few details are different.

Let $x[n]$ be a periodic signal with period of N: $x[\underline{n}+N] = x[n]$

We define the fundamental frequency as $2\pi/N$ in units of rad/sample

In exactly the same way as for CT FS we can decompose the periodic signal $x[n]$ into a sum of complex sinusoids with frequencies that are integer multiples of the fundamental…

The key difference here is that for DT frequencies we can limit our selves to the range 0 to 2π. So the frequencies of interest are: $2\pi k / N, \quad k = 0, 1, 2, \ldots, N-1$

FS for DT Signal

$$\boxed{x[n] = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}, \quad n = 0, 1, 2, \ldots, N-1}$$

Though extendable to other n via periodicity

Similar to how we find the FS coefficients for CT… the DTFS coefficients are

$$\boxed{c_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{j2\pi kn/N}, \quad k = 0, 1, 2, \ldots, N-1}$$

It is easy to verify that $c_{k+N} = c_k$. In other words, the DTFS coefficients themselves have a periodic nature in frequency.

DTFS vs DTFT

$$\boxed{x[n] = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}}$$

$$\boxed{x[n] = \frac{1}{2\pi}\int_{2\pi} X^{\mathrm{f}}(\omega)e^{j\omega n}d\omega}$$

$$\boxed{c_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}}$$

$$\boxed{X^{\mathrm{f}}(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}}$$

Infinite summation… so issues of convergence & truncation

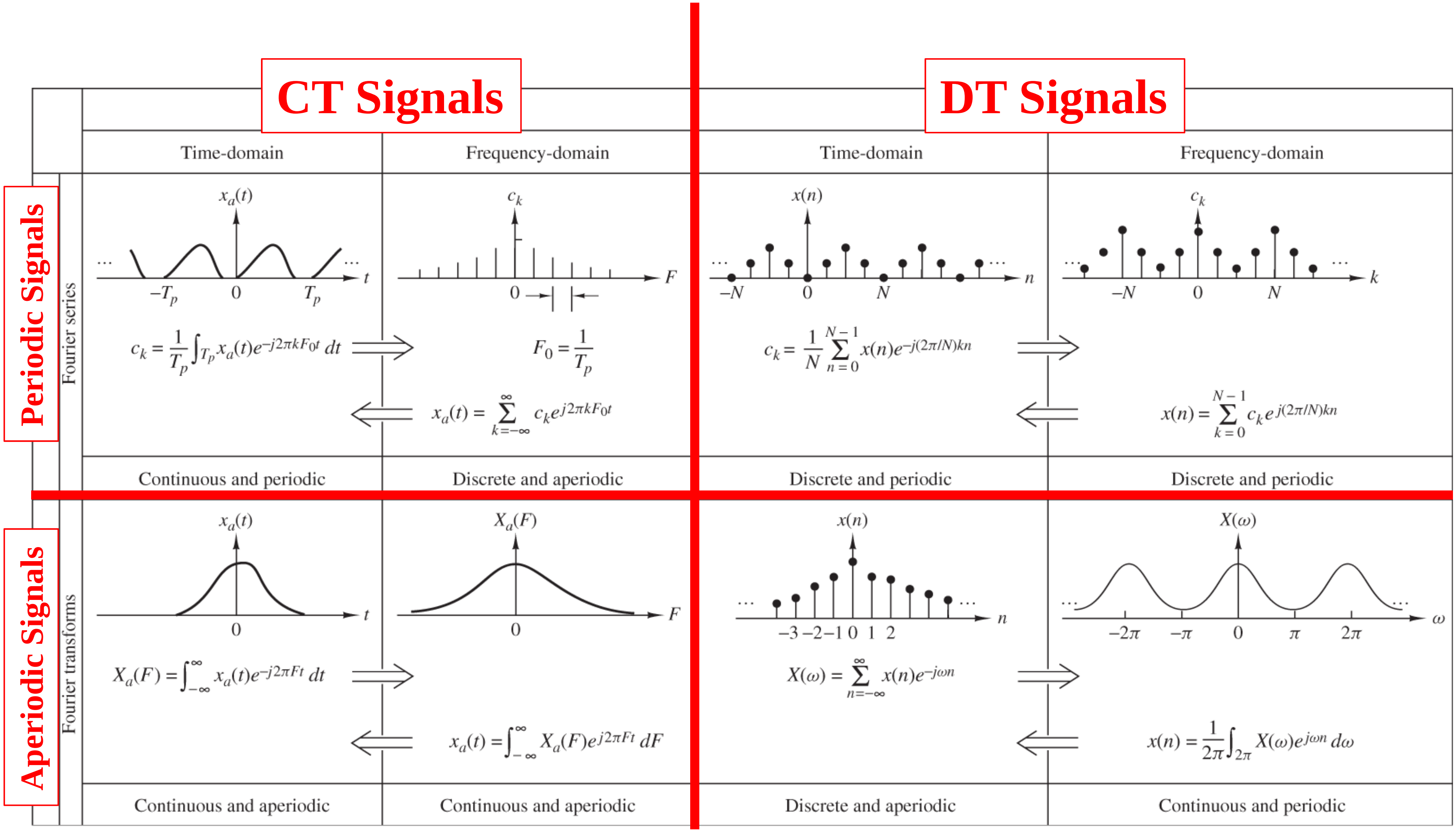


Figure 4.3.1 Summary of analysis and synthesis formulas.

Periodicity w/ "period" α in one domain automatically implies discretization with "spacing" of 1/α in the other domain… and vice versa!

Convergence of DTFT

The equation for finding the DTFT of a signal is $X^{\mathrm{f}}(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$

It involves an infinite sum so there are issues with its convergence….

Recall our study of ROC for ZT and the link between ZT & DTFT!

Define: $X_N^{\mathrm{f}}(\omega) \triangleq \sum_{n=-N}^{N} x[n]e^{-j\omega n}$

Mathematicians have many different ways to characterize convergence… the one we consider first is called "Uniform Convergence". We say that

$X_N^{\mathrm{f}}(\omega)$ converges uniformly to $X^{\mathrm{f}}(\omega)$ when $\lim_{N\to\infty} \left\{ \sup_{\omega} \left| X^{\mathrm{f}}(\omega) - X_N^{\mathrm{f}}(\omega) \right| \right\} = 0$

A "more general" concept of "max"… This roughly says that the "largest" difference between the two keeps getting smaller

Like we saw in our studies of ZT… a sufficient condition is absolute summability of $x[n]$. The DTFT converges uniformly if

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

However… in DSP the class of signals of interest are energy signals ("square summable") and not all of those are absolutely summable…

To allow us to deal with energy signals we have to relax our expectation on the type of convergence we'll accept. We consider "Mean Square Convergence" defined as

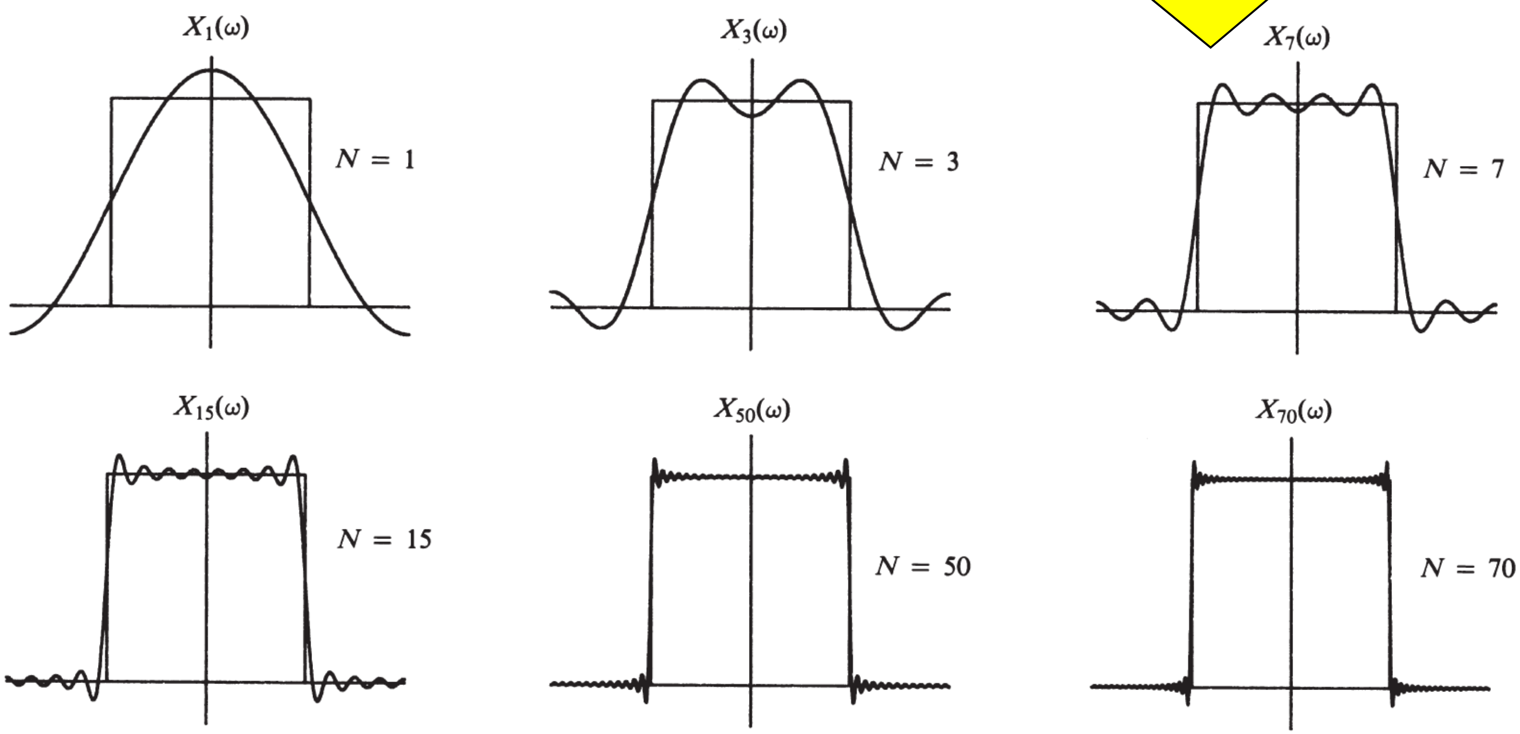
$$\lim_{N\to\infty} \int_{-\pi}^{\pi} \left| X^{\mathrm{f}}(\omega) - X_N^{\mathrm{f}}(\omega) \right|^2 d\omega = 0$$

This says that were are looking for the "area of squared error" to go to zero.

However, the area can go to zero even though the "largest difference" between the two does not go to zero.

Peak of ripple stays same
Area of squared error goes to zero

Example DTFT of sinc function



Relationship of DTFT to ZT

We've already discussed this… but we'll see it again here.

$$X^{z}(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}, \quad ROC: R_2 < |z| < R_1$$

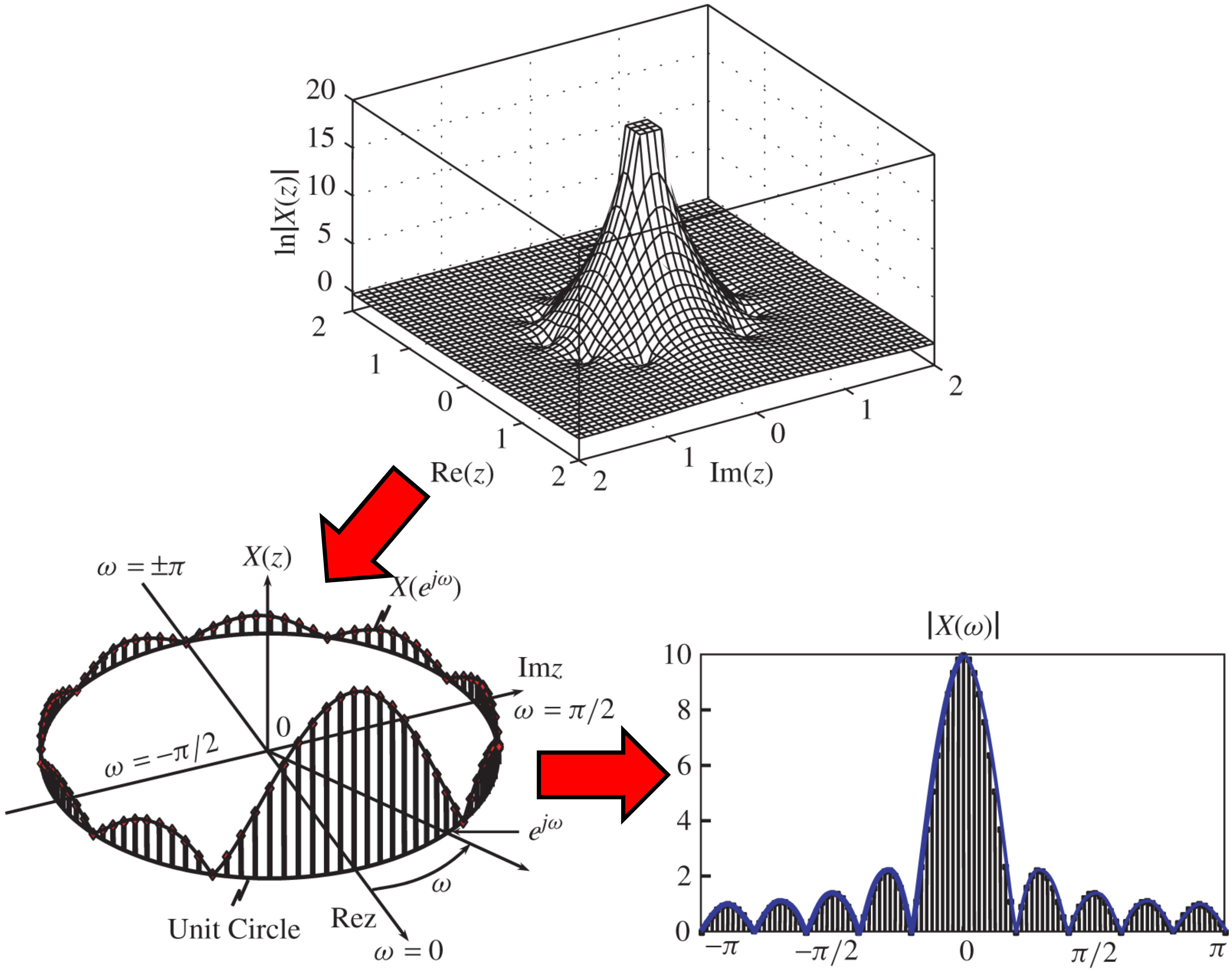
If the ROC contains the UC then we can replace z by its values on the UC: $z \to e^{j\omega}$

$$X^{z}(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \quad \Rightarrow \quad X^{z}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \quad \Rightarrow \quad X^{f}(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

Viewed as a function of ω yields the DTFT

So… If the ROC contains the UC then we get the DTFT by evaluating the ZT on the UC

But… there are some signals w/ DTFT that do not "come from" a ZT. One of those is the sinc function… It has a DTFT but does not have a ZT!



ln|X(z)|
20
15
10
5
0
2
1
0
1
2
Re(z)
2
1
0
1
2
Im(z)
X(z)
ω = ±π
X(e^{jω})
Im z
ω = π/2
0
ω = −π/2
e^{jω}
ω
Unit Circle
Re z
ω = 0
|X(ω)|
10
8
6
4
2
0
−π
−π/2
0
π/2
π

DTFT of Signals w/ Poles on UC

Even though these don't satisfy the "UC in ROC" criteria… if we allow the DTFT to contain delta functions we can get DTFT results for such signals.

Consider the signal $x[n] = u[n]$ whose ZT is $\quad X^z(z) = \dfrac{1}{1 - z^{-1}}$

It has a pole on the UC so clearly the ROC does not contain the UC….

But… in Signals & Systems we said that its DTFT is

$$X^f(\omega) = \frac{1}{1 - e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

Evaluate ZT on UC except at poles

Accounts for pole at z = 1, which is at $\omega = 2\pi k$

Symmetry Properties of the DTFT

Due to similarities between all the Fourier tools… these hold for the other Fourier tools as well

In this section we will allow the signal to be complex valued in general
(Obviously, the DTFT is also in general complex valued)

$$x[n] = x_R[n] + jx_I[n]$$

$$X^{\mathrm{f}}(\omega) = X_R^{\mathrm{f}}(\omega) + jX_I^{\mathrm{f}}(\omega)$$

$$X^{\mathrm{f}}(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \left[x_R[n] + jx_I[n]\right]e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left[x_R[n] + jx_I[n]\right]\left[\cos(\omega n) - j\sin(\omega n)\right]$$

$$= \sum_{n=-\infty}^{\infty} \left[x_R[n]\cos(\omega n) + x_I[n]\sin(\omega n)\right] + j\left[-x_R[n]\sin(\omega n) + x_I[n]\cos(\omega n)\right]$$

Thus.... $X^{\mathrm{f}}(\omega) = X_R^{\mathrm{f}}(\omega) + jX_I^{\mathrm{f}}(\omega)$ with....

$$X_R^{\mathrm{f}}(\omega) = \sum_{n=-\infty}^{\infty} \left[x_R[n]\cos(\omega n) + x_I[n]\sin(\omega n) \right]$$

$$X_I^{\mathrm{f}}(\omega) = -\sum_{n=-\infty}^{\infty} \left[x_R[n]\sin(\omega n) - x_I[n]\cos(\omega n) \right]$$

Similarly.... $x[n] = x_R[n] + jx_I[n]$ with....

$$x_R[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[X_R^{\mathrm{f}}(\omega)\cos(\omega n) - X_I^{\mathrm{f}}(\omega)\sin(\omega n) \right] d\omega$$

$$x_I[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[X_R^{\mathrm{f}}(\omega)\sin(\omega n) - X_I^{\mathrm{f}}(\omega)\cos(\omega n) \right] d\omega$$

We can now use these general results to explore several special cases!

Real Signals

$x_I[n] = 0 \quad x_R[n] = x[n]$

$$X_R^f(\omega) = \sum_{n=-\infty}^{\infty} \left[x_R[n]\cos(\omega n) + \cancel{x_I[n]\sin(\omega n)}\right]$$

$$X_I^f(\omega) = -\sum_{n=-\infty}^{\infty} \left[x_R[n]\sin(\omega n) - \cancel{x_I[n]\cos(\omega n)}\right]$$

$$\boxed{X_R^f(\omega) = \sum_{n=-\infty}^{\infty} x[n]\cos(\omega n)}$$

Even Function because cos is even

$$\boxed{X_R^f(-\omega) = X_R^f(\omega)}$$

$$\boxed{X_I^f(\omega) = -\sum_{n=-\infty}^{\infty} x[n]\sin(\omega n)}$$

Odd Function because sine is odd

$$\boxed{X_I^f(-\omega) = -X_I^f(\omega)}$$

$$\boxed{X^{f*}(\omega) = X^f(-\omega)} \Rightarrow \boxed{\left|X^f(-\omega)\right| = \left|X^f(\omega)\right|} \quad \boxed{\angle X^f(-\omega) = -\angle X^f(\omega)}$$

“Hermitian Symmetry”

Magn is Even

Phase is Odd

Proof: $$X^f(-\omega) = X_R^f(-\omega) + jX_I^f(-\omega)$$

$$= X_R^f(\omega) - jX_I^f(\omega) = X^{f*}(\omega)$$

Real Signals (cont.)

$$x[n] = x_R[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big[\underbrace{X_R^{\mathrm{f}}(\omega)\cos(\omega n)}_{\text{Even = Even x Even}} - \underbrace{X_I^{\mathrm{f}}(\omega)\sin(\omega n)}_{\text{Even = Odd x Odd}}\Big]d\omega$$

So only need to integrate over half the range then x2

$$\boxed{x[n] = \frac{1}{\pi}\int_{0}^{\pi}\Big[X_R^{\mathrm{f}}(\omega)\cos(\omega n) - X_I^{\mathrm{f}}(\omega)\sin(\omega n)\Big]d\omega}$$

Real & Even Signals From the fact that $x[n]$ is real we recall that

$$X_R^{\mathrm{f}}(\omega) = \sum_{n=-\infty}^{\infty} \underbrace{x[n]\cos(\omega n)}_{\textbf{Even = Even x Even}} \qquad X_I^{\mathrm{f}}(\omega) = -\sum_{n=-\infty}^{\infty} \underbrace{x[n]\sin(\omega n)}_{\textbf{Odd = Even x Odd}}$$

$$X_R^{\mathrm{f}}(\omega) = x[0] + 2\sum_{n=1}^{\infty} x[n]\cos(\omega n) \qquad X_I^{\mathrm{f}}(\omega) = 0$$

$$\boxed{X^{\mathrm{f}}(\omega) = X_R^{\mathrm{f}}(\omega) = x[0] + 2\sum_{n=1}^{\infty} x[n]\cos(\omega n)}$$

Even & Real

From the fact that $x[n]$ is real we recall that

$$x[n] = \frac{1}{\pi}\int_0^{\pi} \left[X_R^{\mathrm{f}}(\omega)\cos(\omega n) - X_I^{\mathrm{f}}(\omega)\sin(\omega n) \right] d\omega$$

$$\boxed{x[n] = \frac{1}{\pi}\int_0^{\pi} X_R^{\mathrm{f}}(\omega)\cos(\omega n)\, d\omega}$$

Real & Odd Signals Using a similar argument

$$X_R^{\mathrm{f}}(\omega) = 0 \qquad X_I^{\mathrm{f}}(\omega) = -2\sum_{n=1}^{\infty} x[n]\sin(\omega n)$$

$$\boxed{X^{\mathrm{f}}(\omega) = jX_I^{\mathrm{f}}(\omega) = -2j\sum_{n=1}^{\infty} x[n]\sin(\omega n)}$$

Odd & Imaginary

From the fact that $x[n]$ is real we recall that

$$x[n] = \frac{1}{\pi}\int_0^{\pi}\left[X_R^{\mathrm{f}}(\omega)\cos(\omega n) - X_I^{\mathrm{f}}(\omega)\sin(\omega n)\right]d\omega$$

$$\Rightarrow \quad \boxed{x[n] = -\frac{1}{\pi}\int_0^{\pi} X_I^{\mathrm{f}}(\omega)\sin(\omega n)\,d\omega}$$

Purely Imaginary Signals

$$x_R[n] = 0 \quad x_I[n] = x[n]$$

$$X_R^{f}(\omega) = \sum_{n=-\infty}^{\infty} \left[\cancel{x_R[n]\cos(\omega n)} + x_I[n]\sin(\omega n) \right]$$

$$X_I^{f}(\omega) = -\sum_{n=-\infty}^{\infty} \left[\cancel{x_R[n]\sin(\omega n)} - x_I[n]\cos(\omega n) \right]$$

$$\boxed{X_R^{f}(\omega) = \sum_{n=-\infty}^{\infty} x[n]\sin(\omega n)}$$

Odd Function because sin is odd

$$\boxed{X_R^{f}(-\omega) = -X_R^{f}(\omega)}$$

$$\boxed{X_I^{f}(\omega) = \sum_{n=-\infty}^{\infty} x[n]\cos(\omega n)}$$

Even Function because cos is even

$$\boxed{X_I^{f}(-\omega) = X_I^{f}(\omega)}$$

$$\boxed{x[n] = \frac{1}{\pi}\int_0^{\pi} \left[X_R^{f}(\omega)\sin(\omega n) + X_I^{f}(\omega)\cos(\omega n) \right] d\omega}$$

Imaginary & Even Signals $X_R^f(\omega) = 0$ $X_I^f(\omega) = x[0] + 2\sum_{n=1}^{\infty} x[n]\cos(\omega n)$

$$\boxed{X^f(\omega) = jX_I^f(\omega) = j\left[x[0] + 2\sum_{n=1}^{\infty} x[n]\cos(\omega n)\right]}$$

Even & Imaginary

$$\boxed{x[n] = \frac{1}{\pi}\int_0^{\pi} X_I^f(\omega)\cos(\omega n)\,d\omega}$$

Imaginary & Odd Signals $X_I^f(\omega) = 0$ $X_R^f(\omega) = 2\sum_{n=1}^{\infty} x[n]\sin(\omega n)$

$$\boxed{X^f(\omega) = X_R^f(\omega) = 2\sum_{n=1}^{\infty} x[n]\sin(\omega n)}$$

Odd & Real

$$\boxed{x[n] = \frac{1}{\pi}\int_0^{\pi} X_R^f(\omega)\sin(\omega n)\,d\omega}$$

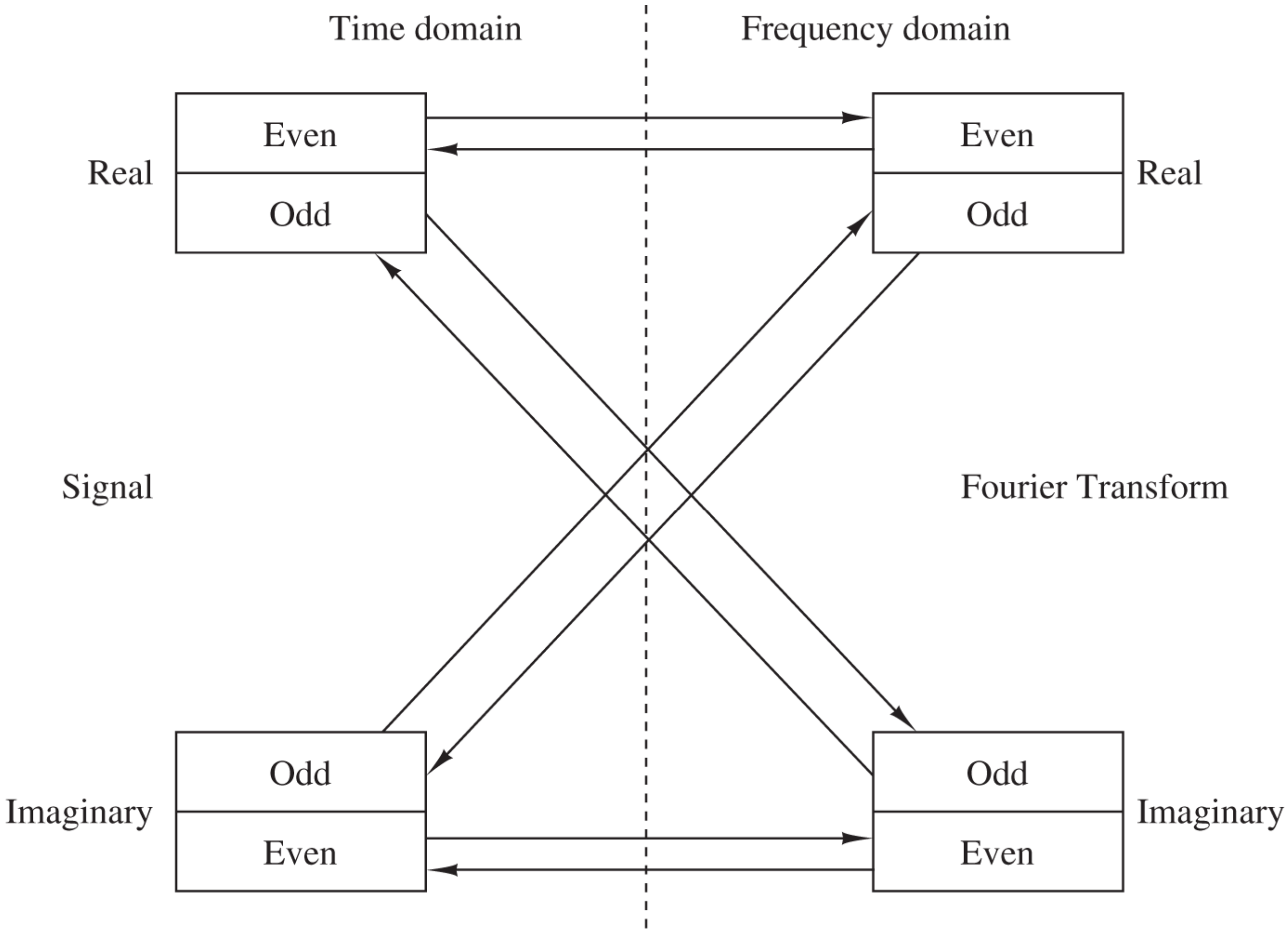


Figure 4.4.2 Summary of symmetry properties for the Fourier transform.

Also see Table 4.4 in the Textbook